LECTURE NOTES F-SINGULARITIES

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Abstract. These are lectures notes for a course on F-singularities given at the CIMAT in the Spring Semester 2024.

Contents

1. Regularity (a crash course)

This is a course about F-singularities and in particular about singularities. In a nutshell, singularities are the absence of regularity. Before defining what a regular ring is, we need the notion of *projective and global dimensions*.

1.1. Projective resolutions and other homological algebra stuff.

Let M be a module over a ring R.[1]

Exercise 1.1. Prove that there is an exact sequence of R-modules

$$0 \longrightarrow K_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where P_0 is free and so projective. Iterate this to obtain an exact sequence

$$0 \longrightarrow K_i \longrightarrow P_{i-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where the P_i's are free. The module K_i is referred to as *a syzygy module*.

Definition 1.1 (Resolutions)**.** An exact sequence

$$\cdots \longrightarrow P_{i+1} \longrightarrow P_i \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is called a *free (resp. projective) resolution* of M if all the P_i's are free (resp. projective). We may denote a projective resolution as $P_\bullet \longrightarrow M \longrightarrow 0$.[2]

Exercise 1.2. Prove that free resolutions always exist, *i.e.* the category of R-modules has "enough projectives."

Definition 1.2 (Projective dimension)**.** The module M is said to have *finite projective dimension* if there is a projective resolution $P_\bullet \longrightarrow M \longrightarrow 0$ such that $P_i = 0$ for all $i \gg 0$. In such case, the *projective dimension* of M is

$$\operatorname{pd} M = \operatorname{pd}_R M := \min\{n \in \mathbb{N} \mid \exists P_\bullet \longrightarrow M \longrightarrow 0 \text{ such that } P_i = 0 \forall i > n\}.$$

If M has not finite projective dimension we write $\operatorname{pd} M = \infty$.

Exercise 1.3. Prove that M is projective iff $\operatorname{pd} M = 0$.

Next lemma is key.

Lemma 1.3. *Suppose that there are two exact sequences of R-modules*

$$0 \longrightarrow K_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

and

$$0 \longrightarrow K'_n \longrightarrow P'_{n-1} \longrightarrow \cdots \longrightarrow P'_0 \longrightarrow M \longrightarrow 0$$

where $1 \leq n \in \mathbb{N}$ and the P_i and P'_i are projective. Then

(a) $K_n \oplus P'_{n-1} \oplus P_{n-2} \oplus \cdots \cong K'_n \oplus P_{n-1} \oplus P'_{n-2} \oplus \cdots$

(b) *K_n is projective iff so is K'_n.*

Proof. Note that (b) follows from (a).[3] The proof of (b) is lengthy and left as an exercise. Hint: Proceed by induction on n. Prove the case $n = 1$ first and then reduce the inductive case to this one. □

[1] All rings are commutative with unity 1.

[2] Over local rings projective modules are free (Kaplansky's theorem). That is, projective modules are locally free. The converse, however, isn't true (unless the module in question is finitely generated).

[3] Observe that for this is absolutely essential to use projectiveness instead of freeness.

It can be used to prove the following.

Exercise 1.4. Let

$$0 \longrightarrow K_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

be an exact sequences where thee P_i's are projective. Prove that

(a) $\operatorname{pd} M \leq n$ iff K_n is projective.
(b) If $\operatorname{pd} M \geq n$ then $\operatorname{pd} K_n = \operatorname{pd} M - n$.

Exercise 1.5. Suppose that R is noetherian and that M is finitely generated. Prove that

$$\operatorname{pd}_R M = \sup\{\operatorname{pd}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\} = \sup\{\operatorname{pd}_{R_\mathfrak{m}} M_\mathfrak{m} \mid \mathfrak{m} \text{ maximal}\}$$

Exercise 1.6. Prove that

$$\operatorname{pd}(M \oplus N) = \max\{\operatorname{pd} M, \operatorname{pd} N\}.$$

The above exercise generalizes as follows.

Exercise* 1.7. Let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of R-modules. Show the following statements.

(a) If two of the modules in the exact sequence have finite projective dimension then so does the third one.
(b) In that case (*i.e.* the three modules have finite projective dimension), then

$$\operatorname{pd} M \leq \max\{\operatorname{pd} M', \operatorname{pd} M''\},$$

(c) and if the inequality is strict then $\operatorname{pd} M'' = \operatorname{pd} M' + 1$.

Definition 1.4 (Minimal free resolution)**.** Let $(R, \mathfrak{m}, \mathcal{k})$ be a noetherian local ring and M a finitely generated R-module. A free resolution $P_\bullet \longrightarrow M \longrightarrow 0$ is said to be *minimal* if

$$\phi_i(P_{i+1}) \subset \mathfrak{m} P_i \quad \forall i \in \mathbb{N}$$

where $\phi_i \colon P_{i+1} \longrightarrow P_i$ is the homomorphism from the free resolution.

Exercise 1.8. In the setup of Definition 1.4, let $K_i := \ker \phi_{i-1}$ for all $i \geq 1$. Prove that $\mu(P_0) = \mu(M)$ and $\mu(P_i) = \mu(K_i)$ for all $i \geq 1$. Here, we let

$$\mu(-) = \dim_{\mathcal{k}} - \otimes_R \mathcal{k}$$

denote the minimal number of generators.

Exercise 1.9. Show that minimal free resolutions exist.

Exercise 1.10. In the setup of Definition 1.4, let $P_\bullet \longrightarrow M \longrightarrow 0$ and $P'_\bullet \longrightarrow M \longrightarrow 0$ be two minimal free resolutions. Show that $\mu(P_i) = \mu(P'_i)$ for all $i \in \mathbb{N}$.

The above two exercises guarantee that the following definition makes sense.

Definition 1.5 (Betti numbers)**.** In the setup of Definition 1.4, the *i-th Betti number* of M is defined as $\beta_i(M) := \mu(P_i)$ where $P_\bullet \longrightarrow M \longrightarrow 0$ is any minimal free resolution.

Remark 1.6. Sometimes people talk about the Betti numbers of $(R, \mathfrak{m}, \mathcal{k})$, in that case, they refer to the Betti numbers of $\mathcal{k}$.

Exercise 1.11. Let $P_\bullet \longrightarrow M \longrightarrow 0$ be a minimal free resolution. Prove that $P_i = 0$ if (and only if) $i > \operatorname{pd} M$. That is,

$$\operatorname{pd} M = \sup\{i \in \mathbb{N} \mid \beta_i(M) \neq 0\}$$

Exercise 1.12. Prove that

$$\beta_i(M) = \dim_{\mathscr{k}} \operatorname{Tor}_i(\mathscr{k}, M), \quad \forall i \in \mathbb{N}.$$

and conclude that

$$\operatorname{pd} M = \sup\{i \in \mathbb{N} \mid \operatorname{Tor}_i(\mathscr{k}, M) \neq 0\} \leq \operatorname{pd} \mathscr{k}.$$

Definition 1.7 (Global dimension)**.** The *global dimension* of a ring R is the supremum of the projective dimensions of finitely generated R-modules.

Corollary 1.8. *The global dimension of a local ring is the projective dimension of its residue field.*

Remark 1.9 (Regular sequences and depth). Recall that a *regular element* $r \in R$ on an R-module M is one for which $\cdot r \colon M \longrightarrow M$ is injective but not surjective. A *regular sequence* $r_1, \dots, r_d \in R$ on M is defined by the following two conditions:

(a) r_1 is regular on M, and
(b) r_i is regular on $M/(r_1, \dots, r_{i-1})M$ for all $i = 2, \dots, d$.

Given an ideal $\mathfrak{a} \subset R$, the *depth of* $\mathfrak{a}$ *on* M, denoted by $\operatorname{depth}_R(\mathfrak{a}, M)$, is the maximal length of a regular sequence on M of elements in $\mathfrak{a}$. When $(R, \mathfrak{m}, \mathscr{k})$ is local, we may write $\operatorname{depth} M = \operatorname{depth}_R M = \operatorname{depth}_R(\mathfrak{m}, M)$. In that case, if $M \neq 0$, we also have:

$$\operatorname{depth} M = \min\{i \in \mathbb{N} \mid \operatorname{Ext}^i(\mathscr{k}, M) \neq 0\}.$$

This formula can be proved as follows (details are left to the reader). First, prove that if $r_1, \dots, r_d \in R$ is a regular sequence on M then

$$\operatorname{Ext}^i_R(\mathscr{k}, M) = \begin{cases} 0 & \text{if } i < d, \\ \operatorname{Hom}_R(\mathscr{k}, M/(r_1, \dots, r_d)M) & \text{if } i = d. \end{cases}$$

This can be proved by induction on d. The base step $d = 0$ is trivial. For the inductive step, consider the exact sequence

$$0 \longrightarrow M \xrightarrow{\cdot r_1} M \longrightarrow M/r_1 M \longrightarrow 0$$

Next, apply the functor $\operatorname{Hom}_R(\mathscr{k}, -)$ to it. Since $r_1 \in \mathfrak{m}$, it acts like 0 on $\mathscr{k}$ and so $\operatorname{Ext}^i_R(\mathscr{k}, \cdot r_1) = 0$. This means that the long exact sequence on Ext's breaks down into exact sequences

$$0 \longrightarrow \operatorname{Ext}^i_R(\mathscr{k}, M) \longrightarrow \operatorname{Ext}^i_R(\mathscr{k}, M/r_1 M) \longrightarrow \operatorname{Ext}^{i+1}_R(\mathscr{k}, M) \longrightarrow 0$$

Since $r_2, \dots, r_d$ is a regular sequence on $M/r_1 M$, we may apply the inductive hypothesis and conclude.

More generally, if $\mathfrak{a} M \neq M$ then

$$\operatorname{depth}_R(\mathfrak{a}, M) = \min\{i \in \mathbb{N} \mid \operatorname{Ext}^i_R(R/\mathfrak{a}, M)\}$$

Exercise 1.13. Here's an important trick to know (particularly, when dealing with Frobenius). Prove that if $r_1, \dots, r_n$ is a regular sequence on M then so is $r_1^{e_1}, \dots, r_d^{e_d}$ for any sequence of exponents $e_1, \dots, e_n \in \mathbb{N}$.

Exercise 1.14. Prove that if r, s is a regular sequence then so is s, r. Find an example of a regular sequence of three elements which is no longer regular after swapping two elements.

Remark 1.10 (Permutations of regular sequences). In general, it turns out that the notion of regular sequence (for more than two elements) is susceptible to the order of the elements. However, this is not the case if we work over a local ring (or in a graded setup). Indeed, given a local ring $(R, \mathfrak{m}, \mathcal{k})$, if $r_1, \ldots, r_d$ is an (M-)regular sequence then so is any permutation of them. Naturally, it suffices to prove this for transpositions of elements. This is something you may try to prove yourself in an elementary fashion (although not so easy). There's, however, a neat (but not so elementary) proof based on a much more general principle. To any sequence of elements we may attach its *Koszul complex*. It turns out that if the sequence is regular its Koszul complex is acyclic. The converse is true over local rings assuming the elements of the sequence belong to the maximal ideal (and in the morally local graded setup). Since the Koszul complex is independent of the permutation of elements in the sequence, then so is the notion of regularity over local rings.

Theorem 1.11 (Auslander–Buchsbaum formula)**.** *In the setup of Definition 1.4, if* $\operatorname{pd} M < \infty$ *then*

$$\operatorname{pd} M + \operatorname{depth} M = \operatorname{depth} R.$$

In particular, if R has finite global dimension it is at most $\operatorname{depth} R$*.*

Proof. We only sketch a proof and leave the details to the reader as an exercise. The proof is an induction on $\operatorname{pd} M$. If $\operatorname{pd} M = 0$ then M is free and so $\operatorname{depth} M = \operatorname{depth} R$. If $\operatorname{pd} M = 1$ then there is an exact sequence

$$0 \longrightarrow R^{\oplus m} \xrightarrow{\phi} R^{\oplus n} \longrightarrow M \longrightarrow 0$$

which we may assume to be minimal, *i.e.* we may assume that the entries of the $n \times m$ R-matrix $\phi\colon R^{\oplus m} \longrightarrow R^{\oplus n}$ are in $\mathfrak{m}$. Consider next the long exact sequence on Ext obtained by applying the functor $\operatorname{Hom}_R(\mathcal{k}, -)$ (write it down yourself). Observe that $\operatorname{Ext}^i_R(\mathcal{k}, R^{\oplus k}) = \operatorname{Ext}^i_R(\mathcal{k}, R)^{\oplus k}$ and that

$$\operatorname{Ext}^i_R(\mathcal{k}, \phi) : \operatorname{Ext}^i_R(\mathcal{k}, R)^{\oplus m} \longrightarrow \operatorname{Ext}^i_R(\mathcal{k}, R)^{\oplus n}$$

is given by the $\mathcal{k}$-matrix obtained by reducing ϕ modulo $\mathfrak{m}$. In particular, $\operatorname{Ext}^i_R(\mathcal{k}, \phi) = 0$ and so there is an exact sequence

$$0 \longrightarrow \operatorname{Ext}^i_R(\mathcal{k}, R)^{\oplus n} \longrightarrow \operatorname{Ext}^i_R(\mathcal{k}, M) \longrightarrow \operatorname{Ext}^{i+1}_R(\mathcal{k}, R)^{\oplus m} \longrightarrow 0$$

From this, we see that $\operatorname{depth} M = \operatorname{depth} R - 1$. This shows the base step of the induction. For the inductive step, suppose $\operatorname{pd} M \geq 2$ and consider an exact sequence

$$0 \longrightarrow N \longrightarrow R^{\oplus m} \longrightarrow M \longrightarrow 0$$

where $\operatorname{pd} N = \operatorname{pd} M - 1$. Use the corresponding long exact sequence on Ext's obtained by applying $\operatorname{Hom}_R(\mathcal{k}, -)$ to find the relationship between the depths of M and N (which is $\operatorname{depth} N = \operatorname{depth} M + 1$). Use the inductive hypothesis to conclude. $\square$

Remark 1.12. It is not difficult to see (using Krull's height theorem and prime avoidance) that every regular sequence can be extended to a system of parameters.[4] In particular,

[4]Indeed, Krull's height theorem let us see that if $r_1, \ldots, r_n \in R$ is a regular sequence then $(r_1, \ldots, r_n)$ has height n. On the other hand, prime avoidance can be used to see that an ideal $(r_1, \ldots, r_n)$ that has height n can be extended to a system of parameters.

depth $R \leq \dim R$.[5] When this equality happens to be an equality one says that $(R, \mathfrak{m}, \mathscr{k})$ is *Cohen–Macaulay*. Thus, a local ring is Cohen–Macaualay if and only if every system of parameters[6] is a regular sequence.

1.2. **Regular local rings.** Let $(R, \mathfrak{m}, \mathscr{k})$ be a noetherian local ring. Then, by Nakayama's lemma, its so-called *embedded dimension*

$$\operatorname{edim} R \coloneqq \mu(\mathfrak{m}) = \dim_{\mathscr{k}} \mathfrak{m} \otimes \mathscr{k} = \dim_{\mathscr{k}} \mathfrak{m}/\mathfrak{m}^2$$

is finite.

Exercise 1.15. Use Krull's ideal theorem to conclude that the embedded dimension is at least the Krull's dimension of the local ring. In particular, noetherian local rings have finite dimension.

Definition 1.13 (Regular local ring)**.** A noetherian local ring $(R, \mathfrak{m}, \mathscr{k})$ is said to be *regular* if the inequality

$$\operatorname{edim} R \geq \dim R$$

is an equality.

Exercise 1.16. Prove that if $(R, \mathfrak{m}, \mathscr{k})$ is a noetherian local ring such that $\mathfrak{m}$ is generated by a regular sequence then it is regular.

The converse of this exercise is also true but a bit harder to prove.

Theorem 1.14. *Let $(R, \mathfrak{m}, \mathscr{k})$ be a regular (noetherian) local ring. Then every set of minimal generators of $\mathfrak{m}$ (aka regular system of parameters) is a regular sequence. In particular, $\operatorname{pd}_R \mathscr{k} = \dim R$.*[7]

This result can be seen as a consequence of the following.

Theorem 1.15. *A regular local ring is an integral domain.*[8]

Recall the following useful, generalized form of prime avoidance.

Lemma 1.16 (Prime avoidance)**.** *Suppose that $\mathfrak{a} \subset \mathfrak{a}_1 \cup \cdots \cup \mathfrak{a}_k$ where all but up to two of the ideals $\mathfrak{a}_i$ are prime. Then $\mathfrak{a} \subset \mathfrak{a}_i$ for some $i = 1, \ldots, k$.*

Lemma 1.17. *Let $(R, \mathfrak{m}, \mathscr{k})$ be a local ring of positive dimension. Then R contains a regular element not in $\mathfrak{m}^2$. That is, there is $r \in \mathfrak{m} \setminus \mathfrak{m}^2$ that avoids all associated primes.*

Proof. Use prime avoidance. □

Sketch of the proof of Theorem 1.15. Set $d = \dim R < \infty$. Let's do induction on d. If $d = 0$, the regularity of R implies that $0 = \dim_{\mathscr{k}} \mathfrak{m}/\mathfrak{m}^2$ and so $\mathfrak{m} = 0$ by Nakayama's lemma. This means that R is a field and we're done.

Assume now that $d > 0$ and that all regular local rings of dimension $< d$ are integral domains. By Lemma 1.17, there is $r \in \mathfrak{m} \setminus \mathfrak{m}^2$ a regular element. Observe that

- R/rR is a local ring whose maximal ideal is generated by $d - 1$ elements (one less than the number of generators of $\mathfrak{m}$), and

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[5] More generally, $\operatorname{depth}(\mathfrak{a}, R) \leq \operatorname{ht} \mathfrak{a}$.

[6] A system of parameters for a local ring $(R, \mathfrak{m}, \mathscr{k})$ is a collection $x_1, \ldots, x_{\dim R}$ such that $\sqrt{(x_1, \ldots, x_{\dim R})} = \mathfrak{m}$. System of parameters always exist.

[7] In particular, regular local rings are Cohen–Macaulay, *i.e.* depth $R = \dim R$.

[8] In fact, they are UFDs and so normal integral domains.

- the dimension of R/rR is $d-1$.

In particular, R/rR is a regular local ring of dimension $d-1$. By the inductive hypothesis, it is an integral domain and so $rR = (r)$ is a prime ideal. Further, observe that $(r) \subset R$ cannot be a minimal prime. Let $\mathfrak{p} \subset R$ be a minimal prime of R that is contained in (r). We're done if we can prove that $\mathfrak{p} = 0$. Let $x \in \mathfrak{p}$, and so $x = yr$ for some $y \in R$. In fact, $y \in \mathfrak{p}$ as $r \notin \mathfrak{p}$. In other words, $\mathfrak{p} = r\mathfrak{p}$. Since $r \in \mathfrak{m}$, Nakayama's lemma yields that $\mathfrak{p} = 0$; as desired. ☐

Corollary 1.18. *Let $(R, \mathfrak{m}, \mathscr{k})$ be a local ring and $r \in \mathfrak{m} \setminus \mathfrak{m}^2$.*[9] *Then, R is regular if and only if r is a regular element and R/rR is regular.*

Summing up, regular local rings have finite global dimension equal to its dimension. It turns out that the converse is also true and it's a deep result due to Auslander–Buchsbaum and Serre. To prove this, we need the following observation.

Exercise 1.17. Let $(R, \mathfrak{m}, \mathscr{k})$ be a local ring and M be a finitely generated R-module. Let $r \in R$ be a regular element on R and on M. Prove that

$$\operatorname{pd}_{R/rR} M/rM = \operatorname{pd}_R M$$

Hint: Show that a minimal free resolution $P_\bullet \longrightarrow M \longrightarrow 0$ becomes a minimal free resolution of M/rM after base change by R/rR. Notice that this is tantamount to the vanishing

$$\operatorname{Tor}_i^R(R/rR, M) = 0, \quad \forall i > 0.$$

But this can be seen from the fact that

$$0 \longrightarrow R \xrightarrow{\cdot r} R \longrightarrow R/rR \longrightarrow 0$$

and

$$0 \longrightarrow M \xrightarrow{\cdot r} M \longrightarrow M/rM \longrightarrow 0$$

are both exact.

We're ready to prove the main result in this section. Please take a moment to appreciate its beauty.

Theorem 1.19 (Auslander–Buchsbaum–Serre)**.** *Let $(R, \mathfrak{m}, \mathscr{k})$ be a local noetherian ring. Then, the following statements are equivalent.*

(a) R is regular (i.e. $\mathfrak{m}$ is generated by a regular sequence)
(b) The global dimension of R is $\dim R$
(c) $\operatorname{pd}_R \mathscr{k}$ is finite.

Proof. It only remains to explain why (c) implies (a). This is an induction on $d := \dim R < \infty$. If $d = 0$, then the Auslander–Buchsbaum formula yields that $\operatorname{pd}_R \mathscr{k} = 0$ and so that $\mathscr{k}$ is a free R-module. Hence, $R = \mathscr{k}$ and we're done.

Let's assume that $d > 0$ and that (c) implies (a) for those local rings of dimension $< d$. Since R is positive dimensional, we can find a regular element $r \in \mathfrak{m} \setminus \mathfrak{m}^2$ and it suffices to prove that the local ring $(R/rR, \mathfrak{m}/rR, \mathscr{k})$ is regular (which has dimension $d-1$). To that end, we can apply the inductive hypothesis and prove that $\operatorname{pd}_{R/rR} \mathscr{k}$ is finite. For this, apply Exercise 1.17. ☐

Exercise 1.18. Prove the following tWo corollaries.

Corollary 1.20. *If $(R, \mathfrak{m}, \mathscr{k})$ is a regular local ring then so is $R_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} R$.*

[9]Note that this is to say that r is part of a minimal set of generators for $\mathfrak{m}$.

Corollary 1.21 (Hilbert's syzygy theorem)**.** *Let $\Bbbk$ be a field. Then, every finitely generated $\Bbbk[x_1,\dots,x_n]$-module has a free resolution of length at most n.*

1.3. **General regular rings.** With the above in place, we can finally define regular rings beyond the local case.

Definition 1.22 (Regular rings of finite dimension)**.** We say that a noetherian ring of finite Krull dimension $\dim R$ is regular if any of the following equivalent conditions hold:

(a) The local ring $R_{\mathfrak{p}}$ is regular for all $\mathfrak{p} \in \operatorname{Spec} R$.
(b) The global dimension of R is at most $\dim R$ (*i.e.* every finitely generated module has projective dimension at most $\dim R$).
(c) R has finite global dimension.

Exercise 1.19. Prove that the above conditions are indeed equivalent.

Definition 1.23 (Regular rings)**.** Let R be a noetherian ring. Then R is said to be regular if $R_{\mathfrak{p}}$ is a regular local ring for all $\mathfrak{p} \in \operatorname{Spec} R$.

Exercise 1.20. Prove that if R is regular then so is $W^{-1}R$ for any multiplicative set $W \subset R$.

Exercise 1.21. Prove that for a regular ring its global dimension equals its dimension.

1.4. **Complete regular rings and the Cohen structure theorems.** Let $(R,\mathfrak{m},\Bbbk)$ be a noetherian local ring. Recall that its completion is the canonical homomorphism

$$R \longrightarrow \hat{R} \coloneqq \varprojlim_n R/\mathfrak{m}^n$$

It turns out that $\hat{R}$ is a noetherian local ring with maximal ideal $\hat{\mathfrak{m}} = \mathfrak{m}\hat{R}$, residue field $\Bbbk$, and dimension $\dim R$. Moreover, $R \longrightarrow \hat{R}$ is a faithfully flat local homomorphism. In particular, R is regular if and only if so is $\hat{R}$.

Remark 1.24. More generally, the completion of an R-module M is the $\hat{R}$-module

$$\hat{M} \coloneqq \varprojlim_n M/\mathfrak{m}^n M.$$

Notice that there is a canonical $\hat{R}$-linear map

$$\hat{R} \otimes_R M \longrightarrow \hat{M}$$

but it may not be an isomorphism. However, it is an isomorphism if M is finitely generated.

Exercise 1.22. Prove that $\operatorname{depth} R = \operatorname{depth} \hat{R}$. In particular, R is Cohen–Macaulay iff so is $\hat{R}$.

Example 1.25. If $R = \Bbbk[x_1,\dots,x_n]/\mathfrak{a}$ and $\mathfrak{m} = (x_1,\dots,x_n)$, then $\hat{R}_{\mathfrak{m}} = \Bbbk[\![x_1,\dots,x_n]\!]/\mathfrak{a}$.

Recall that $(R,\mathfrak{m},\Bbbk)$ is said to be complete if $R \longrightarrow \hat{R}$ is an isomorphism. It turns out that $\hat{R}$ is complete. In fact, every quotient of $\hat{R}$ is a noetherian complete local ring.

Remark 1.26 (Characteristic). Recall that the characteristic of a ring R, say $\operatorname{char} R$, is the only nonnegative integer $n \in \mathbb{N}$ such that $(n) = \ker(\mathbb{Z} \longrightarrow R)$. Note that if R is an integral domain (*i.e.* a field) then $\operatorname{char} R$ is either 0 or a prime number p.

Exercise 1.23. Prove that R contains a field as a subring if and only if $\operatorname{char} R = \operatorname{char} \kappa(\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec} R$. Here $\kappa(\mathfrak{p})$ denotes the residue field of R at $\mathfrak{p}$.

For this reason, those rings that contain a field as a subring are referred to as rings of *equi-characteristic*. If a ring does not contain a field then it is said to have *mixed-characteristic*.

If $(R, \mathfrak{m}, \mathscr{k})$ is a local ring, then it has equicharacteristic iff $\operatorname{char} R = \operatorname{char} \mathscr{k}$. If it is mixed characteristic then $\operatorname{char} \mathscr{k} = p > 0$ but $0 \neq p \in R$.

Suppose that $(R, \mathfrak{m}, \mathscr{k})$ is complete. A complete local subring $(\Lambda, p\Lambda, \mathscr{k}) \subset (R, \mathfrak{m}, \mathscr{k})$ is referred to as a coefficient ring. This entails that $\mathfrak{m} \cap \Lambda = p\Lambda$ and $p = \operatorname{char} \mathscr{k} \geq 0$. There are three cases:

- R has equi-characteristic and so Λ is a field contained in R that maps isomorphically to $\mathscr{k}$.
- R has mixed-caracteristic and $0 \neq p \in R$ is not nilpotent. In that case, $(\Lambda, p\Lambda, \mathscr{k})$ is a complete DVR. We'll referred to this rings as *Cohen rings*.
- R has mixed-caracteristic and $p \in R$ is nilpotent (*i.e.* $\operatorname{char} R = p^n$ for some $n > 1$). In that case, $(\Lambda, p\Lambda, \mathscr{k})$ is an artinian local ring.

Theorem 1.27 (Cohen structure theorem I)**.** *Let $(R, \mathfrak{m}, \mathscr{k})$ be a complete (noetherian) local ring. Then:*

(a) R has a coefficient ring.
(b) There is a surjective homomorphism $\Lambda[\![x_1, \ldots, x_n]\!] \longrightarrow R$ where Λ is either a field or a Cohen ring. Moreover, Λ can be taken as a coefficient ring of R if $p \in R$ isn't nilpotent. In particular, R is a quotient of a regular complete local ring.

Remark 1.28. The most difficult part is to show the existence of a coefficient ring. If $(R, \mathfrak{m}, \mathscr{k})$ has equi-characteristic $p > 0$ and $\mathscr{k}$ is perfect. Then it turns out that

$$\mathscr{k}_0 \coloneqq \bigcap_{e \in \mathbb{N}} R^{p^e}$$

is the only coefficient field of R. Here, $R^{p^e} = \{r^{p^e} \in r \in R\}$.

Theorem 1.29 (Cohen structure theorem II)**.** *Let $(R, \mathfrak{m}, \mathscr{k})$ be a complete regular local ring. Then:*

- *If R has equi-characteristic then $R \cong \mathscr{k}[\![x_1, \ldots, x_n]\!]$.*
- *If R has mixed-characteristic then there is a Cohen ring Λ such that*

$$R \cong \begin{cases} \Lambda[\![x_1, \ldots, x_n]\!] & \text{if } p \in R \text{ is a regular element} \\ \Lambda[\![x_1, \ldots, x_n]\!]/(p - f) \text{ for some } f \in \mathfrak{m}^2 & \text{otherwise.} \end{cases}$$

We say that R is unramified *in the former case.*

Theorem 1.30 (Cohen–Gabber structure theorem III)**.** *Let $(R, \mathfrak{m}, \mathscr{k})$ be a complete local ring that either is equi-characteristic or is an integral domain. Then, there exists a subring $A \subset R$ such that:*

(a) A is a complete local ring,
(b) $A \subset R$ is finite induces an isomorphism on residue fields and is generically étale,
(c) $A \cong \Lambda[\![x_1, \ldots, x_n]\!]$ where Λ is a field or a Cohen ring.

Exercise 1.24. In the setup of Theorem 1.30, show that $(R, \mathfrak{m}, \mathscr{k})$ is Cohen–Macaulay if and only if $A \subset R$ is free (*i.e.* R is a projective A-module). Hint: Use the Auslander–Buchsbaum formula.

Exercise 1.25. Let R be a noetherian equi-characteristic ring. Prove that R is regular iff $\hat{R}_{\mathfrak{p}} \cong \kappa(\mathfrak{p})[\![x_1, \ldots, x_{\operatorname{ht}\mathfrak{p}}]\!]$ for all $\mathfrak{p} \in \operatorname{Spec} R$. Recall that $\kappa(\mathfrak{p}) \coloneqq \mathfrak{p}R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} = \mathscr{K}(R/\mathfrak{p})$ denotes the residue field of $R_{\mathfrak{p}}$.

2. The Frobenius Endomorphism and Kunz's Theorem

From now on unless otherwise stated, we are going to assume that all rings have prime characteristic p. That is, all rings are $\mathbb{F}_p$-algebras. We always use the shorthand notation

$$q \coloneqq p^e.$$

Further, we'll assume that all rings are noetherian. The *Frobenius endomorphism* of a ring R is the homomorphism of $\mathbb{F}_p$-algebras

$$F = F_R \colon R \longrightarrow R, \quad r \mapsto r^p.$$

By iterating, we also have $F^e \colon r \mapsto r^q$ for all $e \in \mathbb{N}$. We let $R^q \subset R$ be the image subring of F^e.

Exercise 2.1. Prove that $F \colon R \longrightarrow R$ is indeed a homomorphism of $\mathbb{F}_p$-algebras. Prove that $\operatorname{Spec} F^e \colon \operatorname{Spec} R \longrightarrow \operatorname{Spec} R$ is the identity.

Exercise 2.2. Prove that R is reduced iff F^e is injective for some/all $e \in \mathbb{N}$.

Exercise 2.3. Recall that a ring R is reduce iff its total ring of fractions $\mathscr{K}(R)$ is a product of fields $K_1 \times \cdots \times K_n$. Then, we may define $\bar{\mathscr{K}}(R)$ as $\bar{K}_1 \times \cdots \times \bar{K}_n$ where $\bar{K}_i$ is an algebraic closure of K_i. Hence $r^{1/q}$ is well-defined in $\bar{\mathscr{K}}(R)$ for all $r \in \mathscr{K}(R)$. Show that

$$R^{1/q} \coloneqq \{r^{1/q} \in \bar{\mathscr{K}}(R) \mid r \in R\} \subset \bar{\mathscr{K}}(R)$$

is a subring that contains R. Moreover, show that $R \subset R^{1/q}$, $F^e \colon R \longrightarrow R$, and $R^q \longrightarrow R$ are isomorphic as R-algebras.

Definition 2.1 (Frobenius powers)**.** Let $\mathfrak{a} \subset R$ be an ideal. Then $\mathfrak{a}^{[q]}$ is the extension ideal of $\mathfrak{a}$ along F^e, and it's called the *e-th Frobenius power of* $\mathfrak{a}$.

Note that if $\theta \colon R \longrightarrow S$ is a homomorphism of rings then there is a commutative diagram

$$\begin{tikzcd} R \arrow[r, "\theta"] \arrow[d, "F^e"'] & S \arrow[d, "F^e"] \\ R \arrow[r, "\theta"] & S \end{tikzcd}$$

Exercise 2.4. Prove that the above diagram is cartesian for all $e \in \mathbb{N}$ if θ is a localization $R \longrightarrow W^{-1}R$. Show that if $\theta \colon R \longrightarrow R/\mathfrak{a}$ is a quotient then the diagram is cartesian iff $\mathfrak{a}^{[q]} = \mathfrak{a}$.

More generally, the following notation is going to be useful.

Notation 2.2 (Frobenius pushforward)**.** Let M be an R-module. We let

$$F^e_* M \coloneqq \{F^e_* m \mid m \in M\}$$

be the R-module defined by the rules $F^e_* m + F^e_* m' = F^e_*(m + m')$ and $rF^e_* m = F^e_* r^q m$. In other words, $F^e_* M$ is the restriction of scalars of M along F^e. Thus, $F^e_* M$ is identical to M as an abelian group but the R-scalar action is being twisted by Frobenius. Likewise, if $M = S$ is an R-algebra then $F^e_* S$ is an R-algebra with the product $(F^e_* s)(F^e_* s') = F^e_*(ss')$. Again, $F^e_* S$ is the exact same thing as S as a ring, what changes is the R-algebra structure.

Exercise 2.5. Prove that R is reduced iff F^e_*R is a faithful R-module for some/all e.

Exercise 2.6. Prove that
$$F^e_*\hat{R} = \widehat{(F^e_*R)}.$$

With the above notation in place, we see that the commutative diagram above induces a ring homomorphism
$$F^e_\theta \colon S \otimes_R F^e_*R \longrightarrow F^e_*S, \quad s \otimes F^e_*r \mapsto F^e_*s^q\theta(r)$$
which is called the *relative Frobenius* of $\theta \colon R \longrightarrow S$.

Exercise 2.7. Prove that $\operatorname{Spec} F^e_\theta$ is a (universal) homeomorphism.

Theorem 2.3 (Kunz's theorem)**.** *Let R be a (noetherian) ring. Then R is regular iff $F^e \colon R \longrightarrow R$ is (faithfully) flat for some/all $e > 0$.*

Remark 2.4 (The socle). Let $(R, \mathfrak{m}, \mathscr{k})$ be a local ring and M be a finitely generated R-module. The *socle* of M is the submodule
$$\operatorname{Soc}(M) \coloneqq \{m \in M : m\mathfrak{m} = 0\} \cong \operatorname{Hom}_R(\mathscr{k}, M) = \operatorname{Ext}^0_R(\mathscr{k}, M).$$
In particular, $\operatorname{depth} M = 0$ iff $\operatorname{Soc} M \neq 0$. Since $\bigcap_{n\in\mathbb{N}} \mathfrak{m}^n M = 0$, it follows that, if $\operatorname{depth} M = 0$, there is $n \in \mathbb{N}$ such that $\operatorname{Soc} M \not\subset \mathfrak{m}^n M$. Let $c \coloneqq \operatorname{depth} R$ and $r_1, \dots, r_c \in R$ be a regular sequence. Set $\mathfrak{a} \coloneqq (r_1, \dots, r_c)$. Observe that $\operatorname{depth}_R R/\mathfrak{a} = 0$. Then, we may find $n \in \mathbb{N}$ such that
$$\operatorname{Soc}_R(R/\mathfrak{a}) \not\subset \mathfrak{m}^n(R/\mathfrak{a}).$$

Lemma 2.5. *Let $(R, \mathfrak{m}, \mathscr{k})$ be a local ring of depth c. Then there is $n \in \mathbb{N}$ such that for all infinite minimal free resolutions*
$$\cdots \longrightarrow R^{\oplus\beta_{i+1}(M)} \xrightarrow{\phi_i} R^{\oplus\beta_i(M)} \longrightarrow \cdots \longrightarrow R^{\beta_0(M)} \longrightarrow M \longrightarrow 0$$
the entries of the matrix ϕ_{c+1} are not all contained in $\mathfrak{m}^n$ (i.e. the image of ϕ_{c+1} is not inside $\mathfrak{m}^n R^{\oplus b_{c+1}} = (\mathfrak{m}^n)^{\oplus b_{c+1}}$). Here $b_i \coloneqq \beta_i(M)$.

Proof. Note that, by the Auslander–Buchsbaum formula, we have that $b_{c+1} \neq 0$ as the resolution has infinite length. This is gonna be important below.

Let $\mathfrak{a} = (r_1, \dots, r_c)$ and n be as in Remark 2.4. In particular, for $N \coloneqq \operatorname{Soc}_R(R/\mathfrak{a})$ we have that $N \not\subset \mathfrak{m}^n N$. Observe that
$$\operatorname{pd}_R R/\mathfrak{a} = c$$
and so
$$\operatorname{Tor}^R_{c+1}(M, R/\mathfrak{a}) = 0.$$
This implies that after base changing the given infinite minimal free resolution we obtain that
$$(R/\mathfrak{a})^{\oplus b_{c+2}} \xrightarrow{\phi_{c+1}/\mathfrak{a}} (R/\mathfrak{a})^{\oplus b_{c+1}} \xrightarrow{\phi_c/\mathfrak{a}} (R/\mathfrak{a})^{\oplus b_c}$$
is exact in the middle. In other words,
$$\ker \phi_c/\mathfrak{a} \subset \operatorname{im} \phi_{c+1}/\mathfrak{a}$$

Now, since the given resolution is minimal, we have that the entries of ϕ_c are all in $\mathfrak{m}$ and so
$$N^{\oplus b_{c+1}} \subset \ker \phi_c/\mathfrak{a}.$$

Thus, putting everything together, if (for the sake of contradiction) the image of ϕ_{c+1} is inside $(\mathfrak{m}^n)^{\oplus b_{c+1}}$, it would follow that

$$N^{\oplus b_{c+1}} \subset \big(\mathfrak{m}^n(R/\mathfrak{a})\big)^{\oplus b_{c+1}}.$$

But, since $b_{c+1} \neq 0$, this implies that

$$N \subset \mathfrak{m}^n(R/\mathfrak{a}),$$

which contradicts the construction of n. Isn't math just so cool? $\square$

Lemma 2.6. *Let $(R, \mathfrak{m}, \mathcal{k})$ be a local ring and M be an R-module. Then $\hat{M}$ is a flat $\hat{R}$-module whenever $\operatorname{Tor}_1^R(M, \mathcal{k}) = 0$ and in particular whenever M is flat.*[10]

Proof. This is a particular case of [Sta23, Tag 0AGW]. $\square$

Lemma 2.7 ([Sta23, Tag 039V])**.** *Let $R \longrightarrow S$ a homomorphism of rings and M be an S-module. If M is a flat R-module and a faithfully flat S-module then $R \longrightarrow S$ is flat.*

Exercise 2.8. Let R be a (noetherian ring). Then, F^e_R is flat as an R-module iff $F^e_* R_\mathfrak{p}$ is flat as an $R_\mathfrak{p}$-module for all $\mathfrak{p} \in \operatorname{Spec} R$. If R is local, then $F^e R$ is flat as an R-module iff $F^e \hat{R}$ is flat as an $\hat{R}$-module. Hint: Apply the two previous lemmas.

Exercise 2.9. Let $R = \mathcal{k}[x_1, \dots, x_d]$ be a polynomial ring over a field $\mathcal{k}$ (or more generally over a ring $\mathcal{k}$ whose Frobenius is free). Let $\{F^e_* \lambda\}_{\lambda \in \Lambda}$ be a $\mathcal{k}$-basis for $F^e_* \mathcal{k} = \mathcal{k}^{1/q}$ (which we may assume contains $F^e_* 1$). Prove that

$$\{F^e_* \lambda x_1^{i_1} \cdots x_d^{i_d}\}_{\lambda \in \Lambda, 0 \leq i_1, \dots, i_d \leq q-1}$$

is an R-basis for $F^e_* R$. Suppose now that Λ is finite so that $F^e_* R$ is free of finite rank. Consider the corresponding dual basis

$$\{\phi_{\lambda, i_1, \dots, i_d} \coloneqq (F^e_* \lambda x_1^{i_1} \cdots x_d^{i_d})^\vee\}_{\lambda \in \Lambda, 0 \leq i_1, \dots, i_d \leq q-1}$$

for $\operatorname{Hom}_R(F^e_* R, R)$. Show that

$$F^e_* R \longrightarrow \operatorname{Hom}_R(F^e_* R, R), \quad F^e_* 1 \mapsto \Phi^e \coloneqq \phi_{1, q-1, \dots, q-1}$$

is an isomorphism. We will be referreing to Φ^e as the *e-th (power of the) Frobenius trace* of R.

Exercise 2.10. Conclude that $F^e_* \mathcal{k}[\![x_1, \dots, x_d]\!]$ is a flat $\mathcal{k}[\![x_1, \dots, x_d]\!]$-module. Show that it is free if $[\mathcal{k}^{1/p} : \mathcal{k}] < \infty$. What about the converse?

Proof of Kunz's theorem. We may assume that $(R, \mathfrak{m}, \mathcal{k})$ is local. Moreover, we may assume that $(R, \mathfrak{m}, \mathcal{k})$ is complete. If R is regular then $R \cong \mathcal{k}[\![x_1, \dots, x_{\dim R}]\!]$ and we're done by Exercise 2.10.

Conversely, suppose that $F^e \colon R \longrightarrow R$ is flat. We want to prove that $\operatorname{pd}_R \mathcal{k} < \infty$. Suppose, for the sake of contradiction that there is an infinite minimal free resolution

$$\cdots \longrightarrow R^{\oplus \beta_{i+1}(\mathcal{k})} \xrightarrow{\phi_i} R^{\oplus \beta_i(\mathcal{k})} \longrightarrow \cdots \longrightarrow R^{\beta_0(\mathcal{k})} \longrightarrow \mathcal{k} \longrightarrow 0$$

That is, $\beta_{c+1}(\mathcal{k}) \neq 0$ for $c = \operatorname{depth} R$. Since $F^e \colon R \longrightarrow R$ is flat for all e, we can base chang this inifinite minimal free resolution to obtain a minimal free resolution

$$\cdots \longrightarrow R^{\oplus \beta_{i+1}(\mathcal{k})} \xrightarrow{\phi_i^{[q]}} R^{\oplus \beta_i(\mathcal{k})} \longrightarrow \cdots \longrightarrow R^{\beta_0(\mathcal{k})} \longrightarrow R/\mathfrak{m}^{[q]} \longrightarrow 0$$

[10] Be cautious, the same can't be said about freenes and hence about projectivity.

where $\phi_i^{[q]}$ is the matrix obtained from ϕ_i by raising its entries to the q-th power. In particular, the entries of $\phi_i^{[q]}$ belong to $\mathfrak{m}^{[q]} \subset \mathfrak{m}^q$ for all i and in particular for $i = \operatorname{depth} R + 1$. This, however, contradicts Lemma 2.5 as $\mathfrak{m}^q \subset \mathfrak{m}^n$ for all $e \gg 0$ such that $q \geq n$. □

2.1. **Relative version of Kunz's theorem.** There is a relative version of Kunz's theorem that goes by the name of Radu–André's theorem. To state it, we need to recall the following definition (the relative notion of F-regularity).

Definition 2.8 (Regular algebras)**.** Let $\theta \colon R \longrightarrow S$ be an R-algebra (where R and S are noetherian). We say that θ is *regular* if it is flat and all its fibers are *geometrically regular*. That is, for all $\mathfrak{p} \in \operatorname{Spec} R$ the $\kappa(\mathfrak{p})$-algebra $S \otimes_R \kappa(\mathfrak{p})$ is noetherian and regular (and noetherian) after any base change by a finitely generated field extension $\mathscr{k}/\kappa(\mathfrak{p})$.[11]

Theorem 2.9 (Radu–André)**.** *Let $\theta \colon R \longrightarrow S$ be an R-algebra. Then, θ is regular iff F^e_θ is (faithfully) flat for all/some $e > 0$.*

On the proof. The most important step is to show that if θ is regular than $S \otimes_R F^e_* R$ is noetherian. With that in place, the result can be obtained from the absolute Kunz theorem and the critere de platitude par fibres. I hope to add more details later on. □

2.2. **Bhatt–Scholze's generalization of Kunz's theorem.** The *(colimit) perfection* of a ring R is

$$R \longrightarrow R_{\mathrm{perf}} \coloneqq \operatorname{colim}(R \xrightarrow{F} R \xrightarrow{F} R \longrightarrow \cdots)$$

We say that R is perfect iff $R \longrightarrow R_{\mathrm{perf}}$ is an isomorphism, *i.e.* Frobenius is an isomorphism on R. Observe that R_{perf} is perfect. Perfect rings are rarely noetherian. In fact, a noetherian perfect ring is a finite product of perfect fields.

Exercise 2.11. Prove that $\operatorname{Spec} R \longrightarrow \operatorname{Spec} R_{\mathrm{perf}}$ is a homeomorphism. Conclude that the perfection of a noetherian local ring has finite dimension.

Theorem 2.10 (Bhatt–Scholze)**.** *Let $(R, \mathfrak{m}, \mathscr{k})$ be a complete local ring (of prime characteristic p). Then its perfection is R_{perf} has finite global dimension.*

Proof. TO BE ADDED. □

This result easily proves Kunz's theorem as follows. Recall that the substantial part of Kunz's theorem is that if $F^e \colon R \longrightarrow R$ is flat for a complete local ring then R is regular, *i.e.* R has finite global dimension. That is, we must show that there is $n \in \mathfrak{n}$ such that for all R-modules one has that

$$\operatorname{Tor}_i^R(\mathscr{k}, M) = 0$$

for all $i \geq n$. To that end, one observes that $R \longrightarrow R_{\mathrm{perf}}$ is faithfully flat and that

$$R_{\mathrm{perf}} \otimes_R \operatorname{Tor}_i^R(\mathscr{k}, M) = \operatorname{Tor}_i^{R_{\mathrm{perf}}}(R_{\mathrm{perf}} \otimes_R \mathscr{k}, R_{\mathrm{perf}} \otimes_R M).$$

Then, we can take n to be the global dimension of R_{perf}, which is finite by Bhatt–Scholze's theorem.

Exercise 2.12. Let $R \longrightarrow S$ be faithfully flat. Show that the global dimension of R is no more than the global dimension of S.

[11]It suffices to ask this for all finite purely inseparable extensions $\mathscr{k}/\kappa(\mathfrak{p})$.

3. F-finiteness and Gabber's Theorem

In studying regularity and therefore singularities, one imposes noetherianity as a basic finiteness condition. In studying F-singularities, one imposes one additional condition. Namely,

Definition 3.1. An $\mathbb{F}_p$-algebra R is *F-finite* if $F^e \colon R \longrightarrow R$ is finite for some/all $e > 0$ (*i.e.* $F^e_* R$ is a finitely generated R-module for all e).

Exercise 3.1. Let R be F-finite. Show that so are its localizations, quotients, and polynomial extensions $R[x_1, \dots, x_n]$. Prove that a field $\mathcal{k}$ is F-finite iff $[\mathcal{k}^{1/q} : \mathcal{k}] < \infty$. Conclude that in such case $\mathcal{k}$-algebras that are either essentially of finite type or complete are F-finite.

Exercise 3.2. F-finiteness has nothing to do with noetherianity. Show that there are noetherian rings that aren't F-finite and vice-versa.

Remark 3.2 (F-finiteness equals kählerianity over $\mathbb{F}_p$)*.* According to Fogarty, $R/\mathbb{F}_{\mathfrak{p}}$ is F-finite iff its R-module of Kähler differentials $\Omega_{R/\mathbb{F}_p}$ is finitely generated, in which case $R/\mathbb{F}_p$ is referred to as kählerian. See [Fog80]. The forward implication is rather trivial and can be left as an exercise for those familiar with Kähler differentials. Although this equivalence is conceptually satisfying, we won't use it in the sequel.

Kunz's theorem takes a much simpler form in that case.

Theorem 3.3 (Kunz's theorem in the F-finite case)**.** *Let R be an F-finite (and noetherian) $\mathbb{F}_p$-algebra. Then, R is regular if and only if $F^e_* R$ is a projective (i.e. locally free of finite rank) R-module. If R is further local, it is regular iff $F^e_* R$ is free of finite rank.*

Exercise 3.3. Show that if R is F-finite then its regular locus is (Zariski-)open.

Exercise 3.4. Let $(R, \mathfrak{m}, \mathcal{k})$ be an F-finite local ring. Show that its completion $R \longrightarrow \hat{R}$ is regular. Hint: Show that $F_{\hat{R}/R}$ is an isomorphism and then conclude using Radu–André's theorem.

Exercise 3.5. Suppose that R is a regular F-finite ring and $\mathfrak{p} \in \operatorname{Spec} R$. Show that the following inclusion of ideals

$$\{r \in R \mid \phi(F^e_* r) \in \mathfrak{p}, \forall \phi \in \operatorname{Hom}_R(F^e_* R, R)\} \supset \mathfrak{p}^{[q]}$$

is an equality.

Definition 3.4 (p-basis)**.** A *(regular) p-basis* for a regular F-finite $R/\mathbb{F}_p$ is a set $x_1, \dots, x_n$ such that

$$F^e_* R = \bigoplus_{0 \leq i_1, \dots, i_n \leq q-1} R F^e_* x_1^{i_1} \cdots x_n^{i_n}.$$

In particular, the rank of $F^e_* R$ is q^n.

Remark 3.5. According to Tyc, a p-basis is the same thing as a *differential basis* (*i.e.* $\Omega_{R/\mathbb{F}_p} = \bigoplus_{i=1}^n R dx_i$). See [Tyc88]. In particular, F-finite fields always admit a p-basis.

Exercise 3.6. Let $R \coloneqq \mathbb{F}_p[x, y]/(x^2 + y^2 - 1)$. Prove that R is regular iff $p \neq 2$. However, R admits a p-basis iff $p \equiv 1 \bmod 4$. Hint: the point is that $-1 = p - 1 \in \mathbb{F}_p$ has a square root if and only $p \equiv 1 \bmod 4$.

Example 3.6. Let $\mathbb{k}$ be an F-finite field. Note that $\mathbb{k}[x_1, \ldots, x_n]$ and $\mathbb{k}[\![x_1, \ldots, x_n]\!]$ both admit a p-basis.

Example 3.7. More generally, a local kählerian regular algebra admits a differentials by a result of Matsumura [**?**]. Then, by Tyc's result, a local F-finite regular ring admits a p-basis. I hope to ellaborate more on this later on.

Remark 3.8 (On restriction, extension, and co-extension of scalars). Let $\theta\colon R \longrightarrow S$ be an R-algebra, and say $f \coloneqq \operatorname{Spec}\theta\colon \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$. This induces three covariant functors $f_*, f^*, f^!$; respectively known as *restriction, extension, and co-extension of scalars*. The restriction of scalars functor f_* goes from the category of S-modules to the one of R-modules. If we have a morphism of S-modules $N \longrightarrow N'$, we can think of it as a morphism of R-modules by restricting scalars along $\theta\colon R \longrightarrow S$, which we denote by $f_*N \longrightarrow f_*N'$. On the other hand, the functor of extension of scalars (aka base change) f^* goes from the category of R-modules to the one of S-modules and it's defined by base change. Namely, if $\phi\colon M \longrightarrow M'$ is a morphism of R-modules then its extension of scalars is the morphism of S-modules

$$f^*M \coloneqq S \otimes_R M \xrightarrow{S\otimes_R\phi\colon s\otimes m \mapsto s\otimes\phi(m)} f^*M' \coloneqq S \otimes_R M'.$$

Finally, the functor $f^!$ of co-extension of scalars goes from R-modules to S-modules and is defined as follows. If $\phi\colon M \longrightarrow M'$ is a morphism of R-modules then $f^!\phi$ is the following morphism of S-modules:

$$\begin{aligned} f^!M \coloneqq \operatorname{Hom}_R(S, M) &\longrightarrow f^!M' \coloneqq \operatorname{Hom}_R(S, M') \\ \mu &\mapsto \phi\circ\mu \end{aligned}$$

It is important to notice that $\operatorname{Hom}_R(S, M)$ is indeed an S-module, where the scalar action of S is given by

$$s\mu \coloneqq \mu\circ(\cdot s)\colon s' \mapsto \mu(ss').$$

Thus, it may be better to denote this as an right action, *i.e.* we may write μs instead of $s\mu$. Note that $\operatorname{Hom}_R(S, R)$ is also an R-module where $r\mu = (\cdot r)\circ\mu\colon s \mapsto r\mu(s)$. Nevertheless, these two linear structures are related as follows:

$$r\mu = \mu\theta(r),$$

from which one may say that the S-module structure determines the R-module one (by restriction of scalars).

These three functors are related by the adjointness:

$$f^* \dashv f_* \dashv f^!$$

Indeed, the co-unit $\epsilon\colon f^*f_* \longrightarrow \mathrm{id}$ is given by

$$\epsilon_N S \otimes_R N \xrightarrow{s\otimes n\mapsto sn} N$$

whereas the unit $\eta\colon \mathrm{id} \longrightarrow f_*f^*$ is given by

$$\eta_M\colon M \xrightarrow{m\mapsto 1\otimes m} S \otimes_R M.$$

Likewise, the co-unit $\mathrm{Tr}\colon f_*f^! \longrightarrow \mathrm{id}$ for the adjointness $f_* \dashv f^!$ is known as *the trace* and is defined as

$$\mathrm{Tr}_M\colon \operatorname{Hom}_R(S, M) \xrightarrow{\mu\mapsto\mu(1)} M$$

whereas its unit $\nu\colon \mathrm{id} \longrightarrow f^! f_*$ is the natural transformation

$$\begin{aligned}\nu_N\colon N &\longrightarrow \mathrm{Hom}_R(S, N)\\ n &\mapsto (s \mapsto sn).\end{aligned}$$

Exercise 3.7. Show that the above pairs of units and co-units define a par of adjointness relations $f^* \dashv f_* \dashv f^!$. That is, show that there are commutative diagrams of natural transformations

$$\begin{tikzcd} f^* \arrow[r, "f^*\eta"] \arrow[dr, "\mathrm{id}"'] & f^* f_* f^* \arrow[d, "\epsilon f^*"] \\ & f^* \end{tikzcd} \qquad \begin{tikzcd} f_* \arrow[r, "\eta f_*"] \arrow[dr, "\mathrm{id}"'] & f_* f^* f_* \arrow[d, "f_*\epsilon"] \\ & f_* \end{tikzcd}$$

defining $f^* \dashv f_*$. Likewise, for $f_* \dashv f^!$, show that we have commutative diagrams of natural transformations

$$\begin{tikzcd} f_* \arrow[r, "f_*\nu"] \arrow[dr, "\mathrm{id}"'] & f_* f^! f_* \arrow[d, "\mathrm{Tr}\, f_*"] \\ & f_* \end{tikzcd} \qquad \begin{tikzcd} f^! \arrow[r, "\nu f^!"] \arrow[dr, "\mathrm{id}"'] & f^! f_* f^! \arrow[d, "f^!\, \mathrm{Tr}"] \\ & f^! \end{tikzcd}\quad .$$

The above means that the natural maps

$$\mathrm{Hom}_S(f^*M, N) \xrightarrow{\psi \mapsto f_*\psi \circ \eta_M} \mathrm{Hom}_R(M, f_*N) \text{ and } \mathrm{Hom}_R(M, f_*N) \xrightarrow{\phi \mapsto \epsilon_N \circ f^*\phi} \mathrm{Hom}_S(f^*M, N)$$

are inverse to each other. Similarly, the natural maps

$$\mathrm{Hom}_S(N, f^!M) \xrightarrow{\psi \mapsto \mathrm{Tr}_M \circ f_*\psi} \mathrm{Hom}_R(f_*N, M) \text{ y } \mathrm{Hom}_R(f_*N, M) \xrightarrow{\phi \mapsto f^!\phi \circ \nu_N} \mathrm{Hom}_S(N, f^!M)$$

are mutually inverse.

Exercise 3.8. Notice that f_* is exact and so that f^* is right-exact whereas $f^!$ is left exact. Observe that f^* is exact iff f_*S is flat but $f^!$ is exact iff f_*S is proyective.

Exercise 3.9. Show that the mapping

$$\mathrm{Hom}(f^*, f^!) \longrightarrow f^!R \coloneqq \mathrm{Hom}_R(S, R), \quad \xi \mapsto \xi_R(1)$$

is a bijection, what's its inverse?

This finishes our general observations on restriction, extension, and co-extension of scalars. How does all this apply to F^e?

Exercise 3.10. Suppose that $R/\mathbb{F}_p$ admits a p-basis (and so it is in particular regular and F-finite), say $x_1, \ldots, x_n$. Let

$$\{\phi_{i_1,\ldots,i_d} \coloneqq (F^e_* x_1^{i_1} \cdots x_d^{i_d})^\vee\}_{0 \le i_1,\ldots,i_d \le q-1}$$

be the corresponding dual basis for $\mathrm{Hom}_R(F^e_*R, R)$. Show that:

(a) The F^e_*R-linear mapping

$$F^e_*R \longrightarrow \mathrm{Hom}_R(F^e_*R, R), \quad F^e_*1 \mapsto \Phi^e \coloneqq \phi_{q-1,\ldots,q-1}$$

is an isomorphism. We will be referring to Φ^e as the *e-th (power of the) Frobenius trace* of R.

(b) The equalities
$$\Phi^{e-1} \circ F_*^{e-1}\Phi^1 = \Phi^e = \Phi^1 \circ F_*\Phi^{e-1}.$$
hold, which justifies to say that Φ^e is the e-th power of $\Phi \coloneqq \Phi^1$. In fact, $\Phi^e = \Phi^a \circ F_*^a\Phi^b$ whenever $e = a + b$.

(c) For all $r \in R$ and $\mathfrak{a}, \mathfrak{b} \subset R$ ideals,
$$(\Phi^e r)(F_*^e\mathfrak{a}) \subset \mathfrak{b} \Longleftrightarrow r \in \mathfrak{b}^{[q]} : \mathfrak{a}.$$

(d) For every ideal $\mathfrak{a} \subset R$ with quotient $R \to A \coloneqq R/\mathfrak{a}$, there is an exact sequence of F_*^eR-modules
$$0 \longrightarrow \mathfrak{a}F_*^eR = F_*^e\mathfrak{a}^{[q]} \longrightarrow F_*^e(\mathfrak{a}^{[q]} : \mathfrak{a}) \xrightarrow{F_*^e r \mapsto (\Phi^e r)/\mathfrak{a}} \operatorname{Hom}_A(F_*^eA, A) \longrightarrow 0$$
which induces an isomorphism of F_*^eA-modules
$$F_*^e\left(\frac{\mathfrak{a}^{[q]} : \mathfrak{a}}{\mathfrak{a}^{[q]}}\right) \xrightarrow{\cong} \operatorname{Hom}_A(F_*^eA, A).$$

(e) If $x_n \in R$ is not a unit then $x_1, \ldots, x_{n-1}$ yields a p-basis on R/x_nR. Furthermore, if $I \subset \{1, \ldots, n\}$ is such that $(x_i \mid i \in I) \neq R$ then $\{x_i\}_{i \in I}$ is a regular sequence on R.

(f) If all the $x_1, \ldots, x_n$ are units then $\dim R = 0$.

(g) More generally, $\dim R \leq n$.

(h) Relabel if necessary so that $x_1, \cdots, x_m$ is such that $S/(x_1, \ldots, x_m)$ is zero dimensional. Show that the canonical map
$$R^{\mathrm{perf}} \coloneqq \bigcap_e R^q \xrightarrow{\subset} S \longrightarrow S/(x_1, \ldots, x_m)$$
is injective.

(i) Conclude that R^{perf} is noetherian and so a product of perfect fields.

Theorem 3.9 (Gabber [Gab04]). *Let $R/\mathbb{F}_p$ be F-finite (and noetherian). Then, there is an F-finite regular ring S admiting a p-basis (and so having) finite dimension such that R is a homomorphic image of S, i.e. there is a quotient $S \twoheadrightarrow R$.*

Main idea of the proof. The proof is constructive. Let $F_*r_1, \ldots, F_*r_n \in F_*R$ be R-generators of F_*R. Equivalently, $r_1, \ldots, r_n \in R$ are generators of R as an R^p-module. Consider the R-algebra
$$S_e \coloneqq R[x_1, \ldots, x_n]/(x_1^q - r_n, \ldots, x_n^q - r_n)$$
Observe that its e-th Frobenius factors as follows
$$\begin{tikzcd} S_e \arrow[r, "F^e"] \arrow[dr, two heads, "\phi_e"'] & S_e \\ & R \arrow[u, hook] \end{tikzcd}$$

Moreover, the map ϕ_e further factors as
$$\begin{tikzcd} S_e \arrow[rr, two heads, "\sigma_e"] \arrow[dr, two heads, "\phi_e"'] & & S_{e-1} \arrow[dl, two heads, "\phi_{e-1}"] \\ & R & \end{tikzcd}$$

Where σ_e acts like Frobenius on R and as the identity on the x's. Therefore, we may take the limit over this inverse system to obtain

$$S := \varprojlim_{e\in\mathbb{N}} S_e$$

Concretely, recall that an element $s \in S$ can be thought of as a sequence $(s_0, s_1, s_2, \ldots)$ where $s_e \in S_e$ and $\sigma_e(s_e) = s_{e-1}$. In particular, we may define (with a slight abuse of notation)

$$x_i := (r_i, x_i, x_i, \ldots) \in S$$

as the constant sequence.[12] Now, note that S_e is an S_{e-1}-algebra and its Frobenius factors as

$$\begin{tikzcd}
S_e \arrow[r, "F"] \arrow[dr, two heads, "\varphi_e"'] & S_e \\
 & S_{e-1} \arrow[u, hook, "\vartheta_e"']
\end{tikzcd}$$

And, moreover, such factorization is compatible with the structural maps of the inverse system defined S. More precisely, we have the following commutative diagram

$$\begin{tikzcd}
 & S_{e+1} \arrow[rr, "F"] \arrow[dl, two heads, bend right, "\sigma_{e+1}"'] \arrow[drr, two heads, "\varphi_{e+1}"] & & S_{e+1} \arrow[dl, two heads, "\sigma_{e+1}"] \\
S_e \arrow[r, "F"] \arrow[dr, two heads, "\varphi_e"'] & S_e & & S_e \arrow[u, hook, "\vartheta_{e+1}"'] \arrow[dll, two heads, bend left, "\sigma_e"] \\
 & S_{e-1} \arrow[u, hook, "\vartheta_e"'] & &
\end{tikzcd}$$

Therefore, by taking the inverse limit, we obtain a commutative diagram

$$\begin{tikzcd}
S \arrow[r, "F"] \arrow[dr, two heads, "\varphi"'] & S \\
 & S \arrow[u, hook, "\vartheta"']
\end{tikzcd}$$

of rings. One readily sees that φ is injective and therefore S is reduced. On the other hand,

$$\{x_1^{i_1} \cdots x_n^{i_n}\}_{0\leq i_1,\ldots,i_n \leq p-1}$$

is a basis for S as an S-module by restriction of scalars along θ. Thus, putting everything together, we see that

$$F_*S = \bigoplus_{0\leq i_1,\ldots,i_n\leq p-1} SF_*x_1^{i_1}\cdots x_n^{i_n}.$$

Thus, we're done if S is noetherian, which is the content of the theorem. This is actually an involved proof. Those interested, can try themselves or read a proof in [MP]. $\square$

Corollary 3.10. *F-finite rings have finite dimension.*

Remark 3.11. Corollary 3.10 was originally obtained by Kunz [Kun76]. However, I understand his proof is flawed due to some equi-dimensionality issues. Nonetheless, there are proofs of Corollary 3.10 that are independent of Gabber's result. I think we'll see one later on.

[12]Caution, not every element of R can be lifted to S. In fact, S isn't in any meaningful way an R-algebra.

Question 3.12 (Noether normalization of F-finite rings). Is an F-finite ring a finite (separable) extension of an F-finite regular ring that admits a p-basis?

Corollary 3.13. *F-finite rings admit a canonical module.*[13] *Namely,*

$$\omega_R \coloneqq \operatorname{Ext}_S^{\dim S - \dim R}(R, S),$$

where $S \twoheadrightarrow R$ is as in Theorem 3.9.

Remark 3.14. By the way, not all excellent rings admit a canonical module, see [**?**]. This is an aspect in which F-finite rings beat excellent ones.

Exercise* 3.11. Prove that R is Cohen–Macaulay iff $\operatorname{Ext}^i_S(R, S) = 0$ for all $i \neq \dim S - \dim R$. So far, we've only defined local Cohen–Macaulay rings. Take the definition of general Cohen–Macaulay as being Cohen–Macaulay at all localizations at prime ideals (so you may reduce to the local case).

I hope the above convinces the reader that F-finite rings are pretty awesome. There's yet another reason why this is the case. Those rings that are pretty awesome for algebraic geometry have already been axiomatized and named, namely *excellent rings*.[14] Their definition is a bit of a mouthful though.

Definition 3.15. A noethering ring is said to be *excellent* if

(a) the completion homomorphism $R_{\mathfrak{p}} \longrightarrow \hat{R}_{\mathfrak{p}}$ is regular for all $\mathfrak{p} \in \operatorname{Spec} R$,
(b) all R-algebras of finite type have open regular loci, and
(c) all R-algebras of finite type are catenary (aka universally catenary).

Theorem 3.16 (Kunz [Kun76])**.** *F-finite rings are excellent. Conversely, a local ring $(R, \mathfrak{m}, \mathscr{k})$ is F-finite if (and only if) it is excellent and $\mathscr{k}$ is F-finite.*

Remark 3.17 (On the proof). The proof is too lengthy to be worthwhile doing here. However, the reader should be able to prove already as an exercise that F-finite rings satisfy the first two properties of excellence; which are referred to as *quasi-excellent*, using Radu–André's theorem for (a). Furthermore, the point is the F-finite property is already a notion of excellence in positive characteristics that is much better to deal with than excellence itself. So for instance, there will be many properties excellent rings have and we'll need that can be obtained directly from F-finiteness. So that's the approach we'll take. A very nice detailed proof can be found in [MP].

4. F-purity, F-splittings, and Fedder's criterion

Ok, here we are, we're ready to introduce our first notion of F-singularity. Let $\theta \colon R \longrightarrow S$ be an R-algebra and $f \coloneqq \operatorname{Spec} \theta \colon \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$. Recall that θ is flat if f^* is an exact functor and it is faithfully flat if it further satisfies any and so all of the following equivalent conditions (for a flat morphism):

(a) For every R-module M, if $M \neq 0$ then $f^*M \neq 0$.
(b) For every sequence $M' \longrightarrow M \longrightarrow M''$ of R-modules, if $f^*M' \longrightarrow f^*M \longrightarrow f^*M''$ is exact then is exact $M' \longrightarrow M \longrightarrow M''$.
(c) The map f is surjective.

[13] Don't worry at all if you don't know what this means. It's a little bit of a mess but we'll get back to it later when we need it. But it's a really important thing worth noting right away.

[14] Feel free to read their Wikipedia entry to glimpse at why.

(d) For every maximal ideal $\mathfrak{m} \in \operatorname{Spec} S$, we have $\mathfrak{m}S \neq S$.

Recall that, since $\operatorname{Spec} F = \mathrm{id}$, flatness and faithfull flatness are the same thing for the Frobenius map. And by Kunz's theorem, they're the same as the regularity of the ring. So what's a natural weakening for faithfull flatness?

Definition 4.1 (Purity and splitness)**.** We say that $\theta\colon R \longrightarrow S$ is *pure* if

$$\eta_M \colon M \longrightarrow f_* f^* M$$

is injective for all R-modules M. We say that $\theta\colon R \longrightarrow S$ is *split* if

$$\mathrm{Tr}_R \colon f_* f^! R \longrightarrow R$$

is surjective.

Remark 4.2. Observe that $\eta_M = \eta_R \otimes M$, and so the purity of $\theta\colon R \longrightarrow S$ means that $\eta_R = \theta\colon R \longrightarrow S$ remains injective after tensoring with any R-module M. On the other hand, note that $\theta\colon R \longrightarrow S$ is split iff there is $\phi \in \operatorname{Hom}_R(S, R)$ such that $\phi(1) = 1$, *i.e.* such that the following diagram commutes

$$\begin{tikzcd} R \arrow[r, "\eta_R=\theta"] \arrow[dr, "\mathrm{id}"'] & S \arrow[d, "\phi"] \\ & R \end{tikzcd}$$

We may refer to any such ϕ as a θ-splitting. In particular, one can tensor this diagram by M to get that $\phi \otimes_R M \colon f_* f^* M \longrightarrow M$ is a splitting of η_M, which forces η_M to be injective. In other words,

$$\text{SPLITNESS} \Longrightarrow \text{PURITY}$$

what about the converse?

Exercise 4.1. Show that in the definition of purity we may have restricted to finitely generated R-modules M only.

Proposition 4.3 (Faithfull flatness implies purity)**.** *If $\theta\colon R \longrightarrow S$ is faitfully flat then it is pure.*

Proof. Let M be an R-module. Then, $\eta_M \colon M \longrightarrow f_* f^* M$ is injective if and only if so is

$$f^*\eta_M \colon f^* M \longrightarrow f^* f_* f^* M$$

However, according to Exercise 3.7, the map

$$\epsilon_{f^*M} \colon f^* f_* f^* M \longrightarrow f^* M$$

is a splitting of $f^*\eta_M$. And so we're done. ☐

Exercise 4.2. Let $\mathbb{Z}_{(p)} \longrightarrow \mathbb{Z}_p$ the canonical homomorphism, *i.e.* p-adic completion. It is faithfully flat and so pure. Show that it is not split, so that purity doesn't imply splitness.

Exercise 4.3. Show that $\theta\colon R \longrightarrow S$ is pure (resp. faithfully flat) iff so is $\theta_{\mathfrak{p}} \colon R_{\mathfrak{p}} \longrightarrow S_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec} R$. Prove that if $\theta\colon R \longrightarrow S$ is split then so are $\theta_{\mathfrak{p}} \colon R_{\mathfrak{p}} \longrightarrow S_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec} R$. What about the converse?

One reason why splitness doesn't have a behavior as nice as the one exhibit by purity is that Hom doesn't commute with flat base change. However,

Exercise 4.4. Let M, N be R-modules and consider a flat R-algebra T. Consider the canonical homomorphism of T-modules

$$T \otimes_R \operatorname{Hom}_R(M, N) \longrightarrow \operatorname{Hom}_T(T \otimes_R M, T \otimes_R N), \quad 1 \otimes \phi \mapsto T \otimes_R \phi.$$

Prove that it is an isomorphism if M is finitely presented Hint: Use a presentation of M to reduce to the free case where it is trivial. Conclude that if $\theta \colon R \longrightarrow S$ is a finite R-algebra such that $\theta_{\mathfrak{p}} \colon R_{\mathfrak{p}} \longrightarrow S_{\mathfrak{p}}$ is split for all $\mathfrak{p} \in \operatorname{Spec} R$ then θ is split.

The next ingredient to elucidate when splitness is the same as purity is the so-called *Matlis Duality.*

4.1. **A quick overview of Matlis duality.** I strongly recommend/urge you to go read the appendix of [ILL+07] if you haven't heard about Matlis Duality and all the stuff around it. But, in a nutshell, here's the deal [Sta23, Tag 08Z9]. Let $(R, \mathfrak{m}, \mathscr{k})$ be an equi-characteristic complete local ring.[15] We consider an *injective hull*

$$E = E(\mathscr{k}) = E_R(\mathscr{k})$$

of $\mathscr{k}$ (as an R-module). By the definition, $E \supset \mathscr{k}$ is an injective R-module such that every non-zero submodule of it intersect $\mathscr{k}$ non-trivially. As a matter of fact, E exists and is unique up non-unique isomorphism.[16] For notation ease, let $\mathscr{N}$ be the (full) subcategory of noetherian R-modules (*i.e.* f.g. R-modules) and $\mathscr{A}$ the one of artinian R-modules. Then, $\mathscr{D} \coloneqq \operatorname{Hom}_R(-, E)$ induces a faithful exact (contravariant) functor

$$\mathscr{D} \colon \mathscr{N} \longrightarrow \mathscr{A}$$

and at the same time

$$\mathscr{D} \colon \mathscr{A} \longrightarrow \mathscr{N}$$

and, moreover, there are natural isomorphisms

$$\mathrm{id}_{\mathscr{N}} \cong \mathscr{D} \circ \mathscr{D} \quad \mathrm{id}_{\mathscr{A}} = \mathscr{D} \circ \mathscr{D}.$$

In other words, $\mathscr{D}$ induces an anti-equivalence between the category of noetherian R-modules and the one of artinian R-modules! That's pretty wild if you ask me. Now, we might come back to it later on with the details. For now, this is all I need you to know. For example, the above relies on the canonical map

$$R \xrightarrow{r \mapsto \cdot r} \operatorname{Hom}_R(E, E)$$

being an isomorphism (telling you that R and E are Matlis dual to one another). Use this to show the following.

Exercise 4.5. Show that, for every finitely generated R-module M, the (finitely generated R-module) $\operatorname{Hom}_R(M, R)$ is Matlis dual to the artinian R-module $E \otimes_R M$.

Later on, we'll see the so-called local duality theorem, which is (at the very least) an extremely powerful tool to compute Matlis duals. For instance, it tells you that the Matlis dual of $\operatorname{Hom}_R(M, \omega_R)$ (which is noetherian) is the *local cohomology* module $H^{\dim R}_{\mathfrak{m}}(M)$ (which is artinian).[17] In particular, ω_R is the Matlis dual of $H^{\dim R}_{\mathfrak{m}}(R)$, and $E = H^{\dim R}_{\mathfrak{m}}(\omega_R)$.

[15]That is, it is a $\mathscr{k}$-algebra.

[16]This is a very general thing—it has nothing to do with $\mathscr{k}$ being the residue field nor R being a complete local ring.

[17]Here, we use that R is complete to say that it has a canonical moodule ω_R, say $\omega_R = \operatorname{Hom}_A(R, A)$ where $A \subset R$ is any noether normalization.

We started our remark assuming that $(R, \mathfrak{m}, \mathscr{k})$ is complete. If we drop this hypothesis, we obtain instead a natural isomorphism

$$\hat{R} \otimes_R - \xrightarrow{\cong} \mathscr{D} \circ \mathscr{D}$$

on $\mathscr{N}$. In particular, we obtain a canonical isomorphism

$$\hat{R} \xrightarrow{\cong} \operatorname{Hom}_R(E, E)$$

which realizes E as an $\hat{R}$-module! In fact, E is an injective hull of $\mathscr{k}$ as an $\hat{R}$-module. Let that sink in. Thus, an injective hull of $\mathscr{k}$ as an $\hat{R}$-module is the same as one as an R-module. This is gonna come up below. Here's an example worth having in mind that illustrates this.

Example 4.4 (Explicit description of injective hulls of residue fields)**.** The injective hull of the residue field, although rather mysterious looking at first, it's often in practice a very explicit object. Let's consider, for example, $R = \mathscr{k}[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$. It turns out that we may take E to be the $\mathscr{k}$-vector space

$$E = E_R(\mathscr{k}) \coloneqq \bigoplus_{0<i_1,\ldots,i_n \in \mathbb{N}} \mathscr{k} \cdot \frac{1}{x_1^{i_1} \cdots x_n^{i_n}} \xleftarrow{u \coloneqq \frac{1}{x_1 \cdots x_n} \leftarrow\!\shortmid 1} \mathscr{k}$$

where the R-linear action is what you expect:

$$x_j \cdot \frac{1}{x_1^{i_1} \cdots x_n^{i_n}} = \begin{cases} \frac{1}{x_1^{i_1} \cdots x_j^{i_j - 1} \cdots x_n^{i_n}} & \text{if } i_j > 1, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, a fix $1/(x_1^{i_1} \cdots x_n^{i_n})$ is killed by all monomials of sufficiently large degree. This is why the above action extends to an $\hat{R} = \mathscr{k}[\![x_1, \ldots, x_n]\!]$-linear! That is, E is also and $\hat{R}$-module. In fact, it equals $E_{\hat{R}}(\mathscr{k})$. Check this:

Exercise 4.6. Verify that the above E is an injective hull of $\mathscr{k}$ over both R and $\hat{R}$.

The element

$$u \coloneqq \frac{1}{x_1 \cdots x_n} \in E$$

is particularly distinguished gentleman. It's called the *socle element* of E. To see why, do the following exercise.

Exercise 4.7. Prove that $\operatorname{Soc} E = \mathscr{k} \cdot u$, where $\mathscr{k} \cdot u$ is the copy of $\mathscr{k}$ inside E.

It's also customary to give E a $\mathbb{Z}$-grading by declaring

$$\deg \frac{1}{x_1^{i_1} \cdots x_n^{i_n}} \coloneqq -(i_1 + \cdots + i_n)$$

Let E_d the direct summand of E of degree d (*e.g.* $E_d = 0$ for all $d > -n$). Letting S_d denote the $\mathscr{k}$-module of polynomials of degree d, the above induces a perfect pairing of $\mathscr{k}$-modules

$$S_d \otimes_{\mathscr{k}} E_{-n-d} \longrightarrow E_{-n} = \mathscr{k} \cdot u \cong \mathscr{k}.$$

The above tell us how to write down an injective hull of $\mathscr{k}$ over $\mathscr{k}[\![x_1, \ldots, x_n]\!]$. But what about other complete $\mathscr{k}$-algebras, *i.e.* quotients of $\mathscr{k}[\![x_1, \ldots, x_n]\!]$? Let $A \coloneqq \mathscr{k}[\![x_1, \ldots, x_n]\!]/\mathfrak{a}$ be a complete local $\mathscr{k}$-algebra. Then, we may take its injective hull of the residue field as

$$E_A(\mathscr{k}) \coloneqq \{\varepsilon \in E = E_{\hat{R}}(\mathscr{k}) \mid \varepsilon\mathfrak{a} = 0\} = \operatorname{Hom}_{\hat{R}}(A, E),$$

which is an (injective) A-module by definition—it is annihilated by $\mathfrak{a}$.

Exercise 4.8. Verify that this is indeed an injective hull of $\mathscr{k}$ over A and show that its socle is $\mathscr{k} \cdot u$. In particular, $u \in E_A(\mathscr{k})$ and we refer to it as the *socle element* of $E_A(\mathscr{k})$.[18]

This is in fact just a particular case of the following.

Theorem 4.5 ([ILL+07, Theorem A.25])**.** *Let $\theta\colon (R, \mathfrak{m}, \mathscr{k}) \longrightarrow (S, \mathfrak{n}, \mathscr{l})$ be a local finite homomorphsim. Then, $E_S(\mathscr{l}) = \operatorname{Hom}_R(S, E_R(\mathscr{k}))$.*

The above explicit description of injective hulls has the virtue of telling us why the following holds.

Exercise 4.9. Let $\mathscr{l}/\mathscr{k}$ be an arbitrary extension of fields and consider the canonical local and flat homomorphism

$$R_{\mathscr{k}} \coloneqq \mathscr{k}[\![x_1, \ldots, x_n]\!] \longrightarrow R_{\ell} \coloneqq \mathscr{l}[\![x_1, \ldots, x_n]\!]$$

obtained as the $(x_1, \ldots, x_n)$-adic completion of the flat canonical homomorphism

$$\mathscr{k}[x_1, \ldots, x_n] \longrightarrow \mathscr{l}[x_1, \ldots, x_n] = \mathscr{l} \otimes_{\mathscr{k}} \mathscr{k}[x_1, \ldots, x_n].$$

Show that

$$E_{R_\ell}(\mathscr{l}) = R_\ell \otimes_{R_{\mathscr{k}}} E_{R_{\mathscr{k}}}(\mathscr{k}),$$

where the socle element of $E_{R_\ell}(\mathscr{l})$ corresponds to 1 tensor the socle element of $E_{R_{\mathscr{k}}}(\mathscr{k})$. More generally, use this to show that, if $\mathfrak{a} \subset R_{\mathscr{k}}$ is an ideal, then

$$E_{R_\ell/\mathfrak{a}R_\ell}(\mathscr{l}) = R_\ell/\mathfrak{a}R_\ell \otimes_{R_{\mathscr{k}}/\mathfrak{a}} E_{R_{\mathscr{k}}/\mathfrak{a}}(\mathscr{k})$$

and that likewise the socle element base changes to the socle element. Observe that $R_{\mathscr{k}}/\mathfrak{a} \longrightarrow R_\ell/\mathfrak{a}R_\ell$ is a (faithfully) flat local homomorphism and so pure. In particular, we obtain a canonical commutative diagram

$$\begin{tikzcd}
E_{R_{\mathscr{k}}/\mathfrak{a}}(\mathscr{k}) \arrow[r, hook] & E_{R_\ell/\mathfrak{a}R_\ell}(\mathscr{l}) \\
\mathscr{k} \arrow[r, hook] \arrow[u, hook, "1 \mapsto u"] & \mathscr{l} \arrow[u, hook, "1 \mapsto u"']
\end{tikzcd}$$

where the diagonal arrows are the socle maps.

Exercise 4.10. Show that a map $\phi \in \operatorname{Hom}_R(E, M)$ is injective if (and only if) $\phi(u) \neq 0$ (where $u \in E$ is the socle element).

4.2. **Back to purity vs splitness.** Using Matlis duality we can see the following.

Proposition 4.6. *Let $\theta\colon (R, \mathfrak{m}, \mathscr{k}) \longrightarrow S$ be an R-algebra such that $(R, \mathfrak{m}, \mathscr{k})$ is complete (and equi-characteristic). Let E the injective hull of $\mathscr{k}$ and $u \in E$ be a socle element. Then, the following statements are equivalent*

(a) θ is pure.
(b) $\eta_E\colon E \longrightarrow S \otimes_R E$ is injective.
(c) $0 \neq 1 \otimes u \in S \otimes_R E$.
(d) θ is split.

[18]Probably we should be saying *a* socle element. A socle element is any generator of the socle and they differ up to multplication by units of $\mathscr{k}$. I guess it's ok to say the socle when it has been chosen in such a specific way.

Proof. We only need to explain why (b) implies (d) (see Exercise 4.10). That is, we gotta show that the trace

$$\mathrm{Tr}_R\colon \mathrm{Hom}_R(S,R) \longrightarrow R$$

is surjective assuming that

$$\eta_E\colon E \longrightarrow S\otimes_R E$$

is injective. Applying the exact contravariant functor $\mathscr{D}(-) = \mathrm{Hom}_R(-,E)$,[19], we obtain that $\mathscr{D}(\eta_E)$ is surjective. However, we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_R(S,R) \arrow[r, "\mathrm{Tr}_R"] \arrow[d, "\sigma\mapsto\sigma\otimes E"'] & R \arrow[d, "\cong\colon 1\mapsto \mathrm{id}_E"] \\
\mathrm{Hom}_R(S\otimes_R E, E) \arrow[r, "\mathscr{D}(\eta_E)"] & \mathrm{Hom}_R(E,E)
\end{tikzcd}$$

where the right-hand vertical arrow is an isomorphism by Matlis duality. We claim that the left-hand vertical arrow is an isomorphism too. Indeed, recall that by $\otimes$-Hom adjointness

$$\mathrm{Hom}_R(S\otimes_R E, E) \underset{(\psi(s)(\varepsilon)\leftarrow\!\shortmid s\otimes\varepsilon)\leftarrow\!\shortmid\psi}{\overset{\phi\mapsto(s\mapsto\phi\circ E\otimes(s:R\longrightarrow S))}{\longleftrightarrow}} \mathrm{Hom}_R(S,\mathrm{Hom}_R(E,E))$$

is an isomorphism. Moreover, the composition

$$\mathrm{Hom}_R(S,R) \xrightarrow{\sigma\mapsto\sigma\otimes E} \mathrm{Hom}_R(S\otimes_R E,E) \xrightarrow{\phi\mapsto(s\mapsto\phi\circ E\otimes(s:R\longrightarrow S))} \mathrm{Hom}_R(S,\mathrm{Hom}_R(E,E))$$

is exactly

$$f^!(R \xrightarrow{\cong\colon 1\mapsto\mathrm{id}_E} \mathrm{Hom}_R(E,E))$$

and so an isomorphism. This proves the proposition. Isn't all this abstract nonsense just pretty? □

Exercise 4.11. In the setup of Proposition 4.6, let M be an R-module. Prove that $\mathscr{D}$ of $\eta_{\mathscr{D}(M)}$ is (up to canonical isomorphism) Tr_M. Hint: in Proposition 4.6 we did the case $M=R$.

Scholium 4.7. *Let $\theta\colon (R,\mathfrak{m},\mathscr{k}) \longrightarrow S$ be a finite R algebra such that R is complete. Then, η_E and Tr_R are Matlis dual to one another. More generally, $\eta_{\mathscr{D}(M)}$ and Tr_M are Matlis dual to one another for all finitely generated R-modules M.*

Corollary 4.8. *Let $\theta\colon R\longrightarrow S$ be a finite R-algebra. If θ is pure then it is split.*

Proof. Since θ is finite, we may assume that R is local (see Exercise 4.4). Since $R\longrightarrow\hat{R}$ is faithfully flat, we assume that R is further complete (see Exercise 4.4). The result then follows from Proposition 4.6. □

Corollary 4.9. *Let $\theta\colon (R,\mathfrak{m},\mathscr{k}) \longrightarrow S$ be an R-algebra and E be an injective hull of $\mathscr{k}$ with socle element $u\in E$. The following statements are equivalent:*

(a) *$\theta\colon R\longrightarrow S$ is pure.*
(b) *$\eta_E\colon E\longrightarrow S\otimes_R E$ is injective (i.e. $0\neq 1\otimes u\in S\otimes_R E$).*
(c) *$\hat{\theta}\colon \hat{R}\longrightarrow\hat{S}$ is pure.*

[19]Whose exactness simply means that E is an injective R-module

Proof. Let's start showing the equivalence between (b) and (c). For this, we use that E is automatically an injective hull of $\mathscr{k}$ as an $\hat{R}$-module. Observe that $\eta_E\colon E \longrightarrow \hat{S} \otimes_{\hat{R}} E$ factors as

$$E \xrightarrow{\eta_E} S \otimes_R E \to \hat{S} \otimes_R E = \hat{S} \otimes_{\hat{R}} E$$

where the second map is E tensor the canonical faithfully flat (and so pure) homomorphism $S \longrightarrow \hat{S}$, whence it is injective. This means that $E \longrightarrow \hat{S} \otimes_{\hat{R}} E$ is injective iff so is $E \longrightarrow S \otimes_R E$.

It remains to explain why (c) implies (a). Assume that $\hat{\theta}$ is pure and so split. Let $\phi \in \operatorname{Hom}_{\hat{R}}(\hat{S}, \hat{R})$ be a splitting. Now, let M be an R-module. Since $R \longrightarrow \hat{R}$ is faithfully flat, to show that $\eta_M\colon M \longrightarrow S \otimes_R M$ is injective, we may do it after base changing it to $\hat{R}$. But $\hat{R} \otimes_R \eta_M$ is injective because it is split by the composition

$$\hat{R} \otimes_R S \otimes_R M \longrightarrow \hat{S} \otimes_R M \xrightarrow{\phi \otimes M} \hat{R} \otimes_R M,$$

where the first map is M tensor the canonical map $\hat{R} \otimes_R S \longrightarrow \hat{S}$, □

4.3. **The purity and splitness of Frobenius.** We made it, here's the key concept in the theory.

Definition 4.10 (F-purity and F-splitness)**.** Let $R/\mathbb{F}_p$ be a noetherian $\mathbb{F}_p$-algebra. We say that R is *F-pure* (resp. F-split) if $F^e\colon R \longrightarrow R$ is pure (resp. split) for some/all $0 \neq e \in \mathbb{N}$. A map $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ such that $\phi(F^e_* 1) = 1$ is referred to as an *F^e-splitting.*

Corollary 4.11. *Both regular rings and F-split rings are F-pure. Further, F-pure rings are reduced.*

Corollary 4.12. *A ring R is F-pure if and only if $R_{\mathfrak{p}}$ is F-pure for all $\mathfrak{p} \in \operatorname{Spec} R$.*

Remark 4.13. The same can't be said about F-splitness. In fact, there are regular rings that aren't F-split; see [**?**]. This is why I don't think you can think of F-splitness as an honest notion of singularity, unless it matches F-purity by some good external reason.

Corollary 4.14. *Suppose that R is either a complete local ring or F-finite. Then, it is F-pure iff it is F-split.*

Corollary 4.15. *Let $(R, \mathfrak{m}, \mathscr{k})$ be a local ring. Then it is F-pure if and only if $\hat{R}$ is F-split.*

Exercise 4.12. Let $\theta\colon R \longrightarrow S$ be a pure (resp. split) R-algebra. Prove that if S is F-pure (resp. F-split) then so is R. Conclude that "direct summands" of regular rings are F-pure.

Exercise 4.13. Working in the setup of Exercise 4.9, prove that $R_{\mathscr{k}}/\mathfrak{a}$ is F-pure (*i.e.* F-split) if and only if so is $R_\ell/\mathfrak{a}R_\ell$. Hint: The forward implication "⇒" follows at once from Exercise 4.12 as $A_{\mathscr{k}} \coloneqq R_{\mathscr{k}}/\mathfrak{a} \longrightarrow A_\ell \coloneqq R_\ell/\mathfrak{a}R_\ell$ is faithfully flat and so pure. The converse is the more interesting part. To prove it, use Exercise 4.9 to chase the socle element along the following commutative diagram

$$\begin{tikzcd}
E_{A_{\mathscr{k}}}(\mathscr{k}) \arrow[r, "\eta"] \arrow[d, hook] & F^e_* A_{\mathscr{k}} \otimes_{A_{\mathscr{k}}} E_{A_{\mathscr{k}}}(\mathscr{k}) \arrow[d, hook, "\theta"] \\
E_{A_\ell}(\ell) \arrow[r, "\eta"] & F^e_* A_\ell \otimes_{A_\ell} E_{A_\ell}(\ell)
\end{tikzcd}$$

where the map θ is injective as it corresponds to the canonical map

$$F^e_* A_{\mathscr{k}} \otimes_{A_{\mathscr{k}}} E_{A_{\mathscr{k}}}(\mathscr{k}) \longrightarrow F^e_* A_\ell \otimes_{F^e_* A_{\mathscr{k}}} (F^e_* A_{\mathscr{k}} \otimes_{A_{\mathscr{k}}} E_{A_{\mathscr{k}}}(\mathscr{k})),$$

which is injective as $A_{\mathscr{k}} \longrightarrow A_\ell$ and so $F^e_* A_{\mathscr{k}} \longrightarrow F^e_* A_\ell$ are pure.

4.4. **Fedder's criterion.** How to know when a ring is F-pure you may wonder? Fedder's criterion is the answer. Let's start with a warm up and let's express Fedder's criterion for rings that admit a p-basis.

Exercise 4.14 (Fedder's criterion for p-bases)**.** Let S be a regular F-finite ring that admits a p-basis; see Exercise 3.10. Let $\mathfrak{a} \subset S$ be an ideal with quotient $R \coloneqq S/\mathfrak{a}$. Let $\mathfrak{p} \subset \mathfrak{a}$ be a prime ideal, which we may think of as a prime ideal of R. In the context of the exact sequence

$$0 \longrightarrow F^e_* \mathfrak{a}^{[q]} \longrightarrow F^e_*(\mathfrak{a}^{[q]} : \mathfrak{a}) \longrightarrow \operatorname{Hom}_R(F^e_* R, R) \longrightarrow 0$$

Show that $\operatorname{Hom}_R(F^e_* R, \mathfrak{p})$ corresponds to

$$F^e_* \left(\frac{(\mathfrak{a}^{[q]} : \mathfrak{a}) \cap \mathfrak{p}^{[q]}}{\mathfrak{a}^{[q]}} \right)$$

Conclude that $R_{\mathfrak{p}}$ is F-pure if and only if $\mathfrak{a}^{[q]} : \mathfrak{a} \not\subset \mathfrak{p}^{[q]}$.

Of course, Fedder's criterion is not very useful unless we can compute $\mathfrak{a}^{[q]} : \mathfrak{a}$, which is in general no joke. There's, however, an important case that isn't terribly hard. Namely,

Lemma 4.16. *Let $\mathfrak{a} \subset S$ be generated by a regular sequence $f_1, \ldots, f_n$ (here, we don't assume that S is regular) that remains regular after any permutation.*[20] *Then, the inclusion*

$$\mathfrak{a}^{[q]} : \mathfrak{a} \supset (f_1^q, \ldots, f_n^q, (f_1 \cdots f_n)^{q-1})$$

is an equality.

Proof. We proceed by induction on n. If $n = 1$, we readily see that $(f)^{[q]} : (f) = (f^{q-1})$ as soon as f is a regular element on S. Indeed, if $sf = hf^q$ then $s = hf^{q-1}$ as f is not a zero-divisor. Assume the result for all $1, \ldots, n-1$. Let $s \in S$ be such that

$$sf_i = \sum_{j=1}^{n} s_{ij} f_j^q, \quad \forall i = 1, \ldots, n$$

for some matrix $s_{ij} \in S$. We want to show that s is divisible bt $(f_1 \ldots f_n)^{q-1}$ module $(f_1^q, \ldots, f_n^q)$. Now, we use Exercise 1.13 to say that $f_1^q, \ldots, f_{n-1}^q, f_n$ is a regular sequence. So, if we look at the n-th equation above, this tells us that $s - s_{nn} f_n^{q-1} \in (f_1^q, \ldots, f_{n-1}^q)$ and so we may assume that s is is divisible by f_n^{q-1}. That is, we're free to replace s by; say, $s' f_n^{q-1}$. Let's look then at the equations

$$s' f_n^{q-1} f_i = \sum_{j=1}^{n} s_{ij} f_j^q, \quad \forall i = 1, \ldots, n-1.$$

Since $f_1^q, \ldots, f_{n-1}^q, f_n^{q-1}$ is a regular sequence, we conclude that $s' f_i - s_{in} f_n \in (f_1^q, \ldots, f_{n-1}^q)$ for all $i = 1, \ldots, n-1$. In particular, $s' f_i \in (f_1^q, \ldots, f_{n-1}^q, f_n)$ for all $i = 1, \ldots, n-1$. Now, here's the thing, let's use that $f_n, f_1^q, \ldots, f_{n-1}^q$ is a regular sequence[21] together with the inductive hypothesis to conclude that

$$s' \in (f_1^q, \ldots, f_{n-1}^q, (f_1 \cdots f_{n-1})^{q-1}, f_n)$$

[20]For instance, we may assume that $n \leq 2$ or well that S is either local or that it is a polynomial ring over a field and the f_i's are homogeneous. See Remark 1.10.

[21]And here's where we need that we need to be able to permute the elements in the regular sequence!

and so

$$s = s' f_n^{q-1} \in (f_1^q, \ldots, f_{n-1}^q, (f_1 \cdots f_n)^{q-1});$$

as required. $\square$

Remark 4.17. I don't know whether Lemma 4.16 still works if we drop the permutability assumption. I think I might've heard people claim it without it but I'm not sure. Certainly, I don't see how to prove it without this hypothesis. Do you? If so, please let me know!

Corollary 4.18 (Fedder's criterion of complete intersections)**.** *Let S be an F-finite regular ring such that $\mathrm{Hom}_S(F^e_* S, S)$ is principally generated by a (Frobenius trace) Φ^e as an $F^e_* S$-module.*[22] *If $\mathfrak{a} \subset S$ is generated by a permutable regular sequence, then $\mathrm{Hom}_R(F^e_* R, R)$ is principally generated by the restriction of $\Phi^e(f_1 \cdots f_n)^{q-1}$ as an $F^e_* R$-module. Moreover, if S admits a p-basis, then R is F-pure at $\mathfrak{p} \subset \mathfrak{a}$ if and only if $(f_1 \cdots f_n)^{q-1} \notin \mathfrak{p}^{[q]}$.*[23]

Exercise 4.15. Let $f \coloneqq x_0^n + \cdots + x_d^n \in S \coloneqq \mathbb{F}_p[x_0, \ldots, x_d]$. Use Fedder's criterion to characterize those triples (p, n, d) for which $R \coloneqq S/f$ is F-pure. Good luck and have fun!

Lemma 4.19 (Colon ideals and flat base change)**.** *Let $\theta \colon R \longrightarrow S$ be a flat algebra and $\mathfrak{a}, \mathfrak{b} \subset R$ be ideals. Then, the inclusion of ideals*

$$(\mathfrak{b} :_R \mathfrak{a})S \subset \mathfrak{b}S :_S \mathfrak{a}S$$

is an equality.

Proof. Since θ is flat, the canonical surjective map $S \otimes_R \mathfrak{a} \longrightarrow \mathfrak{a}S$ is an isomorphism for all ideals $\mathfrak{a} \subset S$. On the other hand, writing $\mathfrak{a} = (a_1, \ldots, a_n)$ (which uses noetherianity) yields an exact sequence

$$0 \longrightarrow \mathfrak{b} \colon \mathfrak{a} \xrightarrow{\subset} R \xrightarrow{r \mapsto (ra_1, \ldots, ra_n)} (R/\mathfrak{b})^{\times n}$$

We may FLAT base change it to obtain an exact sequence

$$0 \longrightarrow S \otimes_R (\mathfrak{b} : \mathfrak{a}) \longrightarrow S \xrightarrow{s \mapsto (sa_1, \ldots, sa_n)} (S/\mathfrak{b}S)^{\times n}$$

which implies what we want. $\square$

Exercise 4.16. Let $\theta \colon (R, \mathfrak{m}) \longrightarrow (S, \mathfrak{n})$ be a local homomorphism and $\mathfrak{a} \subset R$ be an ideal. Show that $\mathfrak{a}^{[q]} : \mathfrak{a} \subset \mathfrak{m}^{[q]}$ if $(\mathfrak{a}S)^{[q]} : \mathfrak{a}S \subset \mathfrak{n}^{[q]}$. Show that the converse holds if θ is flat (and so faithfully flat). Hint: use the previous Lemma 4.19.

Theorem 4.20 (Local Fedder's criterion)**.** *Let $(S, \mathfrak{n}, \mathcal{k})$ be a regular local ring and $\mathfrak{a} \subset \mathfrak{n} \subset S$ be an ideal. Then, the local ring $(R \coloneqq S/\mathfrak{a}, \mathfrak{m} \coloneqq \mathfrak{n}/\mathfrak{a}, \mathcal{k})$ is F-pure if and only if $\mathfrak{a}^{[q]} : \mathfrak{a} \not\subset \mathfrak{n}^{[q]}$.*

Proof. Notice that we may assume that S and so R are complete. Indeed, F-purity is preserved and can be checked after completion (see Corollary 4.15). Furthermore, $\hat{R} = \hat{S}/\mathfrak{a}\hat{S}$, and $\mathfrak{a}^{[q]} : \mathfrak{a} \not\subset \mathfrak{n}^{[q]}$ if and only if $\hat{\mathfrak{a}}^{[q]} : \hat{\mathfrak{a}} \not\subset \hat{\mathfrak{n}}^{[q]}$ (see Exercise 4.16), where $\hat{\mathfrak{a}} \coloneqq \mathfrak{a}\hat{S}$ and $\hat{\mathfrak{n}} = \mathfrak{n}\hat{S}$.

So it suffices to show the case $S = \mathcal{k}[\![x_1, \ldots, x_d]\!]$. If $\mathcal{k}$ and so S were F-finite then we're done as we'd have a p-basis (see Exercise 4.14). To reduce to that case, we consider $S' \coloneqq \mathcal{k}_{\mathrm{perf}}[\![x_1, \ldots, x_n]\!]$ where $\mathcal{k}_{\mathrm{perf}} \supset \mathcal{k}$ is the perfection of $\mathcal{k}$ (which is a perfect and so F-finite field). We can then apply Exercise 4.13 and Exercise 4.16 to $S \longrightarrow S'$ to conclude. $\square$

[22]For instance, if either S admits a p-basis or it is local. It turns out that this is a general abstract property local Gorenstein rings enjoy as we'll see later on. And, as you may expect, regular local rings are Gorenstein.
[23]This last part only needs that $f_1, \ldots, f_n$ is a regular sequence on $R_{\mathfrak{p}}$.

5. Canonical Modules and a First Encounter with F-injectivity

Here's what we want. Let R be a (noetherian as always) ring. We want a finitely generated R-module ω_R such that

$$\operatorname{Hom}_{R_\mathfrak{p}}\big((\omega_R)_\mathfrak{p}, E_{R_\mathfrak{p}}(\kappa(\mathfrak{p})\big) = H^{\dim R_\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p})$$

for all $\mathfrak{p} \in \operatorname{Spec} R$. That is, we want a module that is locally at every point Matlis dual to the top local cohomology module; which is Artinian. When such module ω_R exists, we praise it and refer to it as *a canonical module* over R. If a canonical module exists, it's far from unique. In fact, if ω_R is a canonical module then so is $\omega_R \otimes_R L$ for all invertible[24] R-modules L. However, this is the worst that can happen. For example, over a *normal ring*,[25] we'll see that what is unique is the divisor class associated to ω_R.

As for the existence of canonical modules, this is intimately related to *Gorenstein singularities*. A *Gorenstein ring* can be defined as a Cohen–Macaulay ring where R itself is a canonical module. As a matter of fact, a (noetherian) ring R admits a canonical module if and only if it is the quotient of a finite dimensional Gorenstein ring. As an example, regular rings are Gorenstein.

To be more precise, what one is looking for is a *(normalized) dualizing complex* $\omega_R^\bullet$ which lives in certain derived category of R-modules. The canonical module is just its cohomology modulo in degree $-\dim R$, which needs to be finite so that the canonical module exists. The existence of dualizing complexes is equivalent to being finite over a Gorenstein ring of finite dimension. The Cohen–Macaulay property is engineered exactly so that the dualizing complex collapses into the canonical module, which then becomes a *dualizing module*. That's really it in a nutshell. That's how Cohen–Macaulay and Gorenstein singularities fit into the general landscape of algebraic geometry. Moreover, this is done so that something called *Serre duality* holds—it is the algebro-geomtric analog of Poincaré duality and so it's of paramount importance. We'll see its local version aka local duality.

You see? It's a whole bunch of difficult stuff to grasp. That's life sometimes. Let's give it a shot and let's try to understand this a bit more as it lies at the heart of F-singularity theory too. Ah! And the reason is the following. If R is further F-finite and is *e.g.* local or essentially of finite type over field, then it turns out that it comes equipped with a so-called *Cartier operator*

$$\kappa_R^e \colon F^e_*\omega_R \longrightarrow \omega_R$$

which is arguably the most important object in the whole theory! For example,

Theorem 5.1. *With notation as above, suppose that R is Cohen–Macaulay and normal. Then, R is F-pure if and only if $\kappa_R^e \colon F^e_*\omega_R \longrightarrow \omega_R$ is split.*

So, there's an important weakening to that condition (at least when R is Cohen–Macaulay).

Definition 5.2 (F-injective Cohen–Macaulay rings)**.** With notation as above, R is said to be *F-injective* if $\kappa_R^e \colon F^e_*\omega_R \longrightarrow \omega_R$ is surjective.

Ok. Let's try to understand a bit more of all that. The first thing to know about is local cohomology, which was in part invented by Grothendieck (ofc) for that specific purpose.

[24]Meaning locally free of rank 1.

[25]Which we haven't define yet and I don't know if the reader knows well.

5.1. **A real quick overview on local cohomology.** Let R be a noetherian ring. We've been using the *functor of sections* $\Gamma \coloneqq \operatorname{Hom}_R(R,-)$ quite a bit, which is quite boring as the canonical map $M \to \operatorname{Hom}_R(R,M)$ is an isomorphism. In particular, $\operatorname{Hom}_R(R,M)$ is exact. To make this more interesting cohomologically speaking, we need to look at *sections with support* on some closed subset of $\operatorname{Spec} R$. To this end, let $\mathfrak{a} \subset R$ be an ideal. All we're gonna do depend on $V(\mathfrak{a}) \subset \operatorname{Spec} R$ so we may take $\mathfrak{a}$ to be radical. Anyways, we consider the so-called *functor of sections with support in* $V(\mathfrak{a})$

$$\Gamma_{\mathfrak{a}}(-) \coloneqq \varinjlim_{n\in\mathbb{N}} \operatorname{Hom}_R(R/\mathfrak{a}^n,-).$$

This is also known with the more algebraic name of $\mathfrak{a}$-*torsion functor*, and the reason is that

$$\Gamma_{\mathfrak{a}}(M) = \bigcup_{n\in\mathbb{N}} \{m \in M \mid m\mathfrak{a}^n = 0\} = \{m \in M \mid \operatorname{supp} m \subset V(\mathfrak{a})\} \subset M.$$

Exercise 5.1. Prove that $\Gamma_{\mathfrak{a}}$ is a left exact (covariant) functor but not necessarily exact.

Thus, one defines the i-th *local cohomology functor with support on* $\mathfrak{a}$; denoted by $H^i_{\mathfrak{a}}$, as the i-th derived functor of $\Gamma_{\mathfrak{a}}$. In practice, this means that $H^i_{\mathfrak{a}}(M)$ is the i-th cohomology of the complex $\Gamma_{\mathfrak{a}}(0 \to E^\bullet)$ where $0 \to M \to E^\bullet$ is any injective resolution.

Exercise 5.2. Prove that

$$H^i_{\mathfrak{a}}(-) = \varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^i_R(R/\mathfrak{a}^n,-).$$

Remark 5.3. Observe that $H^i_{\mathfrak{a}}(-) = H^i_{\sqrt{\mathfrak{a}}}(-)$ and moreover

$$\Gamma_I(H^i_{\mathfrak{a}}(M)) = H^i_{\mathfrak{a}}(M).$$

Also, given a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

we obtain a long exact sequence on local cohomology

$$\cdots \to H^{i-1}_{\mathfrak{a}}(M'') \xrightarrow{\delta} H^i_{\mathfrak{a}}(M') \to H^i_{\mathfrak{a}}(M) \to H^i_{\mathfrak{a}}(M'') \xrightarrow{\delta} H^i_{\mathfrak{a}}(M') \to \cdots$$

Example 5.4. Using that

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

is an injective resolution, we readily see that $H^i_{(p)}(\mathbb{Z})$ is zero unless $i = 1$ in which case it equals $\mathbb{Z}[p^{-1}]/\mathbb{Z}$.

Sometimes one may want/need compute directly a local cohomology module. In that case, the Cech complex is pretty useful. Let $r \in R$ (with R noetherian), it's (extended) Cech complex is

$$C(r;R)\colon 0 \to R \to R_r \to 0$$

If we have a sequence $r_1,\dots,r_n \in R$, then its Cech complex is

$$C(r_1,\dots,r_n;R) \coloneqq \bigotimes_{i=1}^n C(r_i;R).$$

If M is an R-module, we also define the Complex

$$C(r_1,\dots,r_n;M) \coloneqq C(r_1,\dots,r_n;R) \otimes_R M.$$

The proof of the following result is beyond the scope of this course.

Theorem 5.5. *The i-th cohomology of the Cech complex $C(r_1, \dots, r_n; M)$ turns out to be $H^i_{(r_1,\dots,r_n)}(M)$.*

With the above, we can do the following computation.

Exercise 5.3. Prove that if $R = \mathscr{k}[x_1, \dots, x_n]$ and $\mathfrak{m} = (x_1, \dots, x_n)$ then

$$H^i_{\mathfrak{m}}(R) = \begin{cases} 0 & \text{if } i \neq d, \\ E_{R_{\mathfrak{m}}}(\mathscr{k}) & \text{otherwise.} \end{cases}$$

NB this makes the polynomial and power series rings pretty awesome and we'd like all rings to be like this. We'll end up defining Gorenstein rings after this.

The following properties are useful and left as an exercise.

Exercise 5.4. Let $\mathfrak{a} \subset R$ be an ideal in a noetherian ring and M be an R-module. Let $\theta\colon R \longrightarrow S$ be an algebra and N be an S-module. Prove that following:

(a) If θ is flat then the canonical morphism $H^i_{\mathfrak{a}}(M) \otimes_R S \longrightarrow H^i_{\mathfrak{a}S}(M \otimes_R S)$ is an isomorphism. That is, $f^* H^i_{\mathfrak{a}}(M) = H^i_{f^*\mathfrak{a}}(f^*M)$ if f is flat.
(b) $H^i_{\mathfrak{a}}(f_* N) = f_* H^i_{\mathfrak{a}S}(N)$.
(c) If R is further local and M is finitely generated then $H^i_{\mathfrak{m}}(M) = H^i_{\hat{\mathfrak{m}}}(\hat{R})$.

5.2. **Cohen–Macaulay and Gorenstein singularities.** With the basics of local cohomology down, we can see what depth and Cohen–Macaulay means in these terms.

Exercise 5.5. Prove that if $\mathfrak{a}M \neq M$ then

$$\operatorname{depth}_R(\mathfrak{a}, M) = \min\{i \in \mathbb{N} \mid H^i_{\mathfrak{a}}(M) \neq 0\}$$

if M is finitely generated. Conclude that a ring R is Cohen–Macaulay if and only if for all $\mathfrak{p} \in \operatorname{Spec} R$ it follows that

$$H^i_{\mathfrak{p}}(R) = 0, \quad \forall i < \operatorname{ht} \mathfrak{p}.$$

Equivalently, if this happens at all maximal ideals only. In particular, a local ring $(R, \mathfrak{m}, \mathscr{k})$ is Cohen–Macaulay if and only if

$$H^i_{\mathfrak{m}}(R) = 0, \quad \forall i < \dim R.$$

Remark 5.6. Let $(R, \mathfrak{m}, \mathscr{k})$ be a local ring. We'll see later that $H^i_{\mathfrak{m}}(R) = 0$ for all $i > \dim R$ and that $H^{\dim R}_{\mathfrak{m}}(R)$ is never zero. This means that a Cohen–Macaulay local ring is one in which all local cohomology groups $H^i_{\mathfrak{m}}(R)$ vanish except $H^{\dim R}_{\mathfrak{m}}(R)$. Observe that a zero dimensional (*i.e.* artinian) ring is always Cohen–Macaulay.

Definition 5.7 (Cohen–Macaulay modules)**.** A finitely generated R- module M is said to be *Cohen–Macaulay* if $\operatorname{depth} M_{\mathfrak{p}} = \dim M_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec} R$. If $\dim M_{\mathfrak{p}} = \operatorname{ht} \mathfrak{p}$, M is said to be *maximal Cohen–Macaulay.*

Definition 5.8 (Gorenstein rings)**.** A Cohen–Macaulay ring R is said to be *Gorenstein* if $H^{\operatorname{ht}\mathfrak{p}}_{\mathfrak{p}}(R)$ is an injective hull of $\kappa(\mathfrak{p})$ as an $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \operatorname{Spec} R$.[26] In other words, $\mathscr{D}(R_{\mathfrak{p}}) = H^{\operatorname{ht}\mathfrak{p}}_{\mathfrak{p}}(R)$ for all $\mathfrak{p} \in \operatorname{Spec} R$.

Remark 5.9. We've seen that a local ring $(R, \mathfrak{m}, \mathscr{k})$ is Cohen–Macaulay iff $\operatorname{Ext}^i_R(\mathscr{k}, R) = 0$ for all $i \neq \dim R$. It is further Gorenstein if $\operatorname{Ext}^{\dim R}_R(\mathscr{k}, R) = \mathscr{k}$. That's perhaps the definition we've seen before.

[26]Note that Gorenstein rings are first and foremost Cohen–Macaulay (and noetherian!).

Example 5.10. From Exercise 5.3, one readily sees that regular rings are Gorenstein. Next, we see that complete intersections are also Gorenstein.

Exercise 5.6. Let $r_1, \dots, r_n \in (R, \mathfrak{m}, \mathcal{k})$ be a regular sequence. Prove that R is Gorenstein iff so is $R/(r_1, \dots, r_n)$.

Exercise 5.7 (Depth and exact sequences)**.** Let $(R, \mathfrak{m}, \mathcal{k})$ be a local ring and consider the short exact sequence of R-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0.$$

Prove the following:

(a) $\operatorname{depth} M \geq \min\{\operatorname{depth} M', \operatorname{depth} M''\}$
(b) $\operatorname{depth} M'' \geq \min\{\operatorname{depth} M, \operatorname{depth} M' - 1\}$
(c) $\operatorname{depth} M' \geq \min\{\operatorname{depth} M, \operatorname{depth} M'' + 1\}$

What are Gorenstein rings good for you may ask? We'll, let's see that next.

5.3. **Local duality.** Local duality is roughly at the very least an amazing way to understand/calculate Matlis duality. Let's see how grandiose it is for Gorenstein rings. Local duality is pretty much the Poincaré duality of singularities. It's proof, for now, is beyond the scope of this course.

Theorem 5.11 (Local duality over Gorenstein singularities)**.** *Let $(R, \mathfrak{m}, \mathcal{k})$ be a Gorenstein local ring of dimension d. Then, for all $0 \leq i \leq d$, the functors $H^i_{\mathfrak{m}}$ and $\mathscr{D} \circ \operatorname{Ext}^i_R(-, R)$ are naturally isomorphic on the category of finitely generated modules.*[27]*. In particular, if R is complete and M is a finitely generated R-module, the Matlis dual of $\operatorname{Ext}^i_R(M, R)$ is naturally isomorphic to $H^{d-i}_{\mathfrak{m}}(M)$ (for all $0 \leq i \leq d$).*

As a first corollary, one obtains the following.

Exercise 5.8. Let M be a finitely generated module over a local ring $(R, \mathfrak{m}, \mathcal{k})$. Show that $H^i_{\mathfrak{m}}(M)$ is an artinian R-module. Hint: reduce to the complete case. Then, use the Cohen structure theorem to reduce to the regular and so Gorenstein case. From here, use local and Matlis duality.

Now, let G be a Gorenstein ring of finite dimension (which is equidimensional as it is Cohen–Macaulay) and suppose that R is a finite G-algebra. In particular, we can write down a factorization

$$G \twoheadrightarrow A \hookrightarrow R$$

so that R is a finite extension of A, which is itself a quotient of G. By the Cohen structure theorems, this can always be setup if R is complete. Likewise, by Gabber's theorem, also if R is F-finite. Suppose that R and so A are equidimensional (*e.g.* Cohen–Macaulay).

First, we set

$$\omega_G \coloneqq G$$

and then

$$\omega_A \coloneqq \operatorname{Ext}_G^{\dim G - \dim A}(A, \omega_G)$$

which is an A-module. Finally, set

$$\omega_R \coloneqq \operatorname{Hom}_A(R, \omega_A).$$

[27] Reall that $\mathscr{D} \coloneqq \operatorname{Hom}_R(-, E)$ where $E = E_R(\mathcal{k})$

This definition may look quite strange at first. However, it makes total sense under the light of *Grothendieck's duality*. Unfortunately, this is a topic that we have no time to cover as it'd imply to delve into derived categories and so on. This is the right language to make all this very natural and satisfying. As an application of Grothendieck duality, one could see the following.

Proposition 5.12. *With notation as above, let $\mathfrak{p} \in \operatorname{Spec} R$. Then, there is a natural isomorphism of functors*
$$\mathscr{D} \circ \operatorname{Hom}_{R_\mathfrak{p}}(-, (\omega_R)_\mathfrak{p}) \cong H^{\operatorname{ht}\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}$$
on finitely generated $R_\mathfrak{p}$-modules. In particular, $\omega_{\hat{R}_\mathfrak{p}} := \hat{R}_\mathfrak{p} \otimes \omega_R$ is naturally Matlis dual to $H^{\dim R_\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p})$.

In other words, we've succeded in constructing *canonical modules* as mentioned in the introduction, at least for equidimensional finite algebras over Gorentein rings. If R is further Cohen–Macaulay, a canonical module becomes a *dualizing module*:

Theorem 5.13 (Local Duality over Cohen–Macaulay singularities)**.** *Let $(R, \mathfrak{m}, \mathscr{k})$ be a Cohen–Macaulay local ring of dimension d that is a finite algebra over a Gorenstein ring (e.g. complete or F-finite). Let ω_R be the corresponding canonical module. Then, for all $0 \leq i \leq d$, there is a natural isomorphism of functors*
$$\mathscr{D} \circ \operatorname{Ext}^i_R(-, \omega_R) \cong H^{d-i}_\mathfrak{m}$$
over finitely generated R-modules. In particular, if R is complete and M is a finitely generated R-module, then $\operatorname{Ext}^i_R(M, \omega_R)$ and $H^{d-i}_\mathfrak{m}(M)$ are naturally Matlis dual to one another.

Remark 5.14. It turns out that, with R as before, R is Cohen–Macaulay if and only if R is a flat/projective/locally-free A-module and
$$\operatorname{Ext}^i_G(A, G) = 0, \quad \forall i \neq \dim G - \dim A$$
which in turns means that A is Cohen–Macaulay.

Exercise 5.9. Conclude that if R is Cohen–Macaulay then the dualizing module ω_R is a maximal Cohen–Macaulay module.

Exercise 5.10. Suppose that R is an artinian local ring (and so Cohen–Macaulay). Use local duality to show that the canonical map
$$M \longrightarrow \operatorname{Hom}_R(\operatorname{Hom}_R(M, \omega_R), \omega_R)$$
is an isomorphism for all f.g. R-modules M. In particular, the canocial map $R \longrightarrow \operatorname{Hom}_R(\omega_R, \omega_R)$ is an isomorphism.

To recap and what you need to take home, when R happens to be an equi-dimensional finite algebra over a Gorenstein ring of finite dimension (*e.g.* complete or F-finite), we define its canonical module as before. The first cool thing about it (thanks to Grothendieck duality and local duality for Gorenstein singularities) is that there are local natural isomorphisms at every $\mathfrak{p} \in \operatorname{Spec} R$
$$\mathscr{D} \circ \operatorname{Hom}_{R_\mathfrak{p}}(-, (\omega_R)_\mathfrak{p}) \cong H^{\operatorname{ht}\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}$$
In particular, $\omega_{\hat{R}_\mathfrak{p}} := \hat{R}_\mathfrak{p} \otimes \omega_R$ is naturally Matlis dual to $H^{\dim R_\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p})$. So, in the complete case, ω_R is the Matlis dual of the top local cohomology module. If R is local then all its canonical modules are isomorphic, so in that case we may talk about of the canonical module.

Moreover, if R is further Cohen–Macaulay, then the canonical module acquires new powers known as local duality: there is a natural isomorphism

$$\mathscr{D} \circ \operatorname{Ext}^i_{R_{\mathfrak{p}}}(-, (\omega_R)_{\mathfrak{p}}) \cong H^{\operatorname{ht}\mathfrak{p}-i}_{\mathfrak{p}R_{\mathfrak{p}}}$$

on finitely generated $R_{\mathfrak{p}}$-modules for all $\mathfrak{p} \in \operatorname{Spec} R$.

5.4. **The Cartier operator and F-injectivity.** Let's look at the functoriality of canonical modules under finite extensions. Let $R \subset S$ be a finite extension such that R admits a canonical module as before. Let $f\colon \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ be the induced map as usual. By this Grothendieck duality abstract nonsense we've been using shamelessly (I may add the details at some point but don't hold your breath), it follows that $f^!\omega_R$ is a canonical module over S. But what if we already had defined a canonical module ω_S over S? At the very least, we can say that

$$f^!\omega_R \cong \omega_S \otimes_S L$$

for some invertible S-module L. In particular, if $(S, \mathfrak{n})$ and so $(R, \mathfrak{m})$ are local, we can say that

$$f^!\omega_R \cong \omega_S$$

In particular, in this local case, the trace map can be written as

$$\operatorname{Tr}_{\omega_R}\colon f_*\omega_S \longrightarrow \omega_R$$

Observe that we have a commutative diagram

$$\begin{tikzcd} S \arrow[rr, "1\mapsto \operatorname{Tr}_{\omega_R}"] \arrow[dr, "1\mapsto \mathrm{id}"'] & & \operatorname{Hom}_S(f_*\omega_S, \omega_R) \\ & \operatorname{Hom}_S(\omega_S, \omega_S = f^!\omega_R) \arrow[ur, "\cong"'] & \end{tikzcd}$$

of S-modules. We'll see below that Exercise 5.25 $S \longrightarrow \operatorname{Hom}_S(\omega_S, \omega_S)$ is an isomorphism when S has the propoerty of being $\mathbf{S}_2$, which is to say that principal ideals have no embedded primes (*e.g.* Cohen–Macaulay, normal). In that case, we'll have that

$$S \xrightarrow{1\mapsto \operatorname{Tr}_{\omega_R}} \operatorname{Hom}_S(f_*\omega_S, \omega_R)$$

is an isomorphism.

As the reader may verify themselves, applying $\mathscr{D}$ to $\operatorname{Tr}_{\omega_R}$ yields the map:

$$H^d_{\mathfrak{m}}(f)\colon H^d_{\mathfrak{m}}(R) \longrightarrow H^d_{\mathfrak{m}}(S) = f_*H^d_{\mathfrak{n}}(S)$$

where d is the common dimension of R and S. Thus, the completion of $\operatorname{Tr}_{\omega_R}$ is the Matlis dual of the canonical map $H^d_{\mathfrak{m}}(R) \longrightarrow f_*H^d_{\mathfrak{n}}(S)$.

There is one more non-local case where it is possible to construct a trace $\operatorname{Tr}_{\omega_R}\colon f_*\omega_S \longrightarrow \omega_R$. Namely, if R and S are (essentially) of finite type over some field, say $\Bbbk \subset \ell$; respectively, and $\ell/\Bbbk$ is finite. This is Grothendieck duality again. In that case, it must localize (up to isomorphism) to the local traces we had above.

We want to apply the above to the Frobenius map $F^e\colon R \longrightarrow R$, assuming R is further reduced and F-finite.[28] As before, we want the following to hold:

$$F^{e,!}\omega_R \cong \omega_R$$

[28]This is arguably the main reason why F-finiteness matters.

This can be guaranteed either if R is local or essentially of finite type over an F-finite field. But, whenever that's the case, we get the following map

$$\kappa_R^e \coloneqq \mathrm{Tr}_{\omega_R} \colon F_*^e \omega_R \longrightarrow \omega_R$$

which we'll referred to as the *Cartier operator* of R. It follows that

$$F_*^e R \xrightarrow{1 \mapsto \kappa_R^e} \mathrm{Hom}_R(F_*^e \omega_R, \omega_R)$$

is an isomorphism whenever satisfies Serre's condition $\mathbf{S}_2$; as we'll see below.

Exercise 5.11. Prove that $\kappa_R^e = \kappa_R^a \circ F_*^a \kappa_R^b$ whenever $a + b = e$. Writting $\kappa_R \coloneqq \kappa_R^1$, this justifies the notation of κ_R^e as the e-th power of κ_R.

Exercise 5.12. With notation as above, suppose that R is further Gorenstein. Show that $\mathrm{Hom}_R(F_*^e R, R)$ is a free $F_*^e R$ module of rank 1. A generator is often referred to as a Frobenius trace.[29]

We know what κ_R^e has to be formally-locally around any point. Indeed, $\hat{R}_\mathfrak{p} \otimes \kappa_R^e = \kappa_{\hat{R}_\mathfrak{p}}^e$ is Matlis dual to the canonical map

$$H_\mathfrak{p}^{\operatorname{ht} \mathfrak{p}}(F^{e,\#}) \colon H_\mathfrak{p}^{\operatorname{ht} \mathfrak{p}}(R) \longrightarrow F_*^e H_\mathfrak{p}^{\operatorname{ht} \mathfrak{p}}(R)$$

at every point $\mathfrak{p} \in \operatorname{Spec} R$.

Exercise 5.13. Let $R = \mathscr{k}[\![x_1, \ldots, x_n]\!]$ where $\mathscr{k}$ is an F-finite field. Show that κ_R^e is (up to isomorphism) the map Φ^e obtained from the p-basis $x_1, \ldots, x_n$. See Exercise 3.10.

Definition 5.15 (F-injective ring)**.** A ring R is said to be *F-injective* if the canonical map

$$H_\mathfrak{p}^i(F^{e,\#}) \colon H_\mathfrak{p}^i(R) \longrightarrow F_*^e H_\mathfrak{p}^i(R)$$

is injective for all $\mathfrak{p} \in \operatorname{Spec} R$ and all i.

Exercise 5.14. Show that R is F-injective if and only if so is $R_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} R$.

Exercise 5.15. Prove that an F-pure ring is F-injective.

Exercise 5.16. Assume that R admits a canonical module and further a Cartier operator $\kappa_R^e \colon F_*^e \omega_R \longrightarrow \omega_R$; as before. Prove that if R is F-injective then κ_R^e is surjective. Show that converse holds if R is further Cohen–Macaulay. Conclude that if R is further Gorenstein then F-purity and F-injectivity are equivalent notions.

Exercise 5.17. Assume that R admits a canonical module and further a Cartier operator $\kappa_R^e \colon F_*^e \omega_R \longrightarrow \omega_R$; as before. Prove that if R is F-pure then

$$\kappa_{\hat{R}_\mathfrak{p}}^e = \hat{R}_\mathfrak{p} \otimes_R \kappa_R^e$$

is split for all $\mathfrak{p} \in \operatorname{Spec} R$. What about the converse?

Exercise* 5.18. Let $(R, \mathfrak{m}, \mathscr{k})$ be a local Cohen–Macaulay ring with Cartier operator $\kappa_R^e \colon F_*^e \omega_R \longrightarrow \omega_R$. Let $r \in R$ be a regular element. Prove that if R/r is F-injective then so is R.

[29]It's not true that $\mathrm{Hom}_R(F_*^e R, R)$ is a free $F_*^e R$ module of rank 1 for an arbitrary Gorenstein ring R. This is literally the hypothesis $F^{e!} \omega_R \cong \omega_R$, so be careful.

5.5. **Normal rings and divisors.** Normal rings improve upon reduced rings. To see why, I need you do to the following.

Exercise 5.19 ($\mathbf{R}_0 + \mathbf{S}_1$ characterization of reducedness)**.** Prove that a ring R is reduced (*i.e.* it has no nonzero nilpotents) if and only if the following two conditions happen:

(a) If $\mathfrak{p} \in \operatorname{Spec} R$ has height 0 (*i.e.* it is a minimal prime) then $R_\mathfrak{p}$ is regular. This condition is referred to as *regularity in codimension* 0.
(b) If $\mathfrak{p} \in \operatorname{Spec} R$ then $\operatorname{depth} R_\mathfrak{p} \geq \min\{1, \operatorname{ht} \mathfrak{p}\}$. This condition is referred to as *Serre's first condition*.

In general, one says:

Definition 5.16. Let R be a ring, M be an R-module, and $k \in \mathbb{N}$. We say that R satisfies the $\mathbf{R}_k$ (resp. $\mathbf{G}_k$) condition (aka *regularity in codimension* k) iff $R_\mathfrak{p}$ is regular (resp. Gorenstein) whenever $\operatorname{ht} \mathfrak{p} \leq k$. We may simply say that R is/has $\mathbf{R}_k$. Likewise, we say that M satisfies the condition $\mathbf{S}_k$ condition (aka *Serre's k-th condition*) if

$$\operatorname{depth} M_\mathfrak{p} \geq \min\{k, \operatorname{ht} \mathfrak{p}\}$$

for all $\mathfrak{p} \in \operatorname{Spec} R$.

Remark 5.17. Notice that if $M \neq 0$ is an R-module that satisfies $\mathbf{S}_k$, then M is supported at all codimension $\leq k-1$ points, *i.e.* $M_\mathfrak{p} \neq 0$ for all $\mathfrak{p} \in \operatorname{Spec} R$ of height $\leq k-1$. In particular, if M satisfies $\mathbf{S}_1$ and vanishes in codimension 0 then M is zero. Likewise, M satisfies $\mathbf{S}_2$ and vanishes in codimension 1 then M is zero.

Exercise 5.20. Show the following statements:

(a) R is regular (resp. Cohen–Macaulay) iff it satisfies $\mathbf{R}_k$ (resp. $\mathbf{S}_k$) for all k.
(b) R satisfies $\mathbf{S}_1$ iff it has no embedded primes.
(c) R satisfies $\mathbf{S}_2$ iff every principal ideal (r) has no embedded primes.
(d) R satisfies $\mathbf{S}_k$ iff every ideal $\mathfrak{a}$ generated by $\operatorname{ht} \mathfrak{a} \leq k-1$ elements has no embedded primes (this is to say that $\mathfrak{a}$ is unmixed).
(e) Conclude that a ring is Cohen–Macaulay if and only if every ideal generated by its height many elements is unmixed, *i.e.* all its associated primes have the same height (*i.e.* no embedded primes and equi-dimension).

Exercise 5.21. Let R be a ring admitting a canonical module ω_R. Prove that ω_R satisfies $\mathbf{S}_1$. *Hint:* Reduce to the local positive dimensional case. Then do the usual trick of using Lemma 1.17.

Exercise 5.22. Let R be an $\mathbf{S}_1$ ring admitting a canonical module ω_R. Prove that ω_R satisfies $\mathbf{S}_2$. *Hint:* Reduce to the local positive dimensional case. Then do the usual trick of using Lemma 1.17.

Exercise 5.23. Let R be a one-dimensional Cohen–Macaulay ring admitting a canonical module ω_R. Show that the canonical map

$$M \longrightarrow \operatorname{Hom}_R(\operatorname{Hom}_R(M, \omega_R), \omega_R)$$

is an isomorphism iff M is an f.g. R-module of depth 1 (*i.e.* M is a maximal Cohen–Macaulay module). In particular, $R \longrightarrow \operatorname{Hom}_R(\omega_R, \omega_R)$ is an isomorphism.

Exercise 5.24. Let R be an $\mathbf{S}_1$ ring admitting a canonical module ω_R and M be an f.g. R-module. Prove that $\mathrm{Hom}_R(M, \omega_R)$ satisfies $\mathbf{S}_2$. *Hint:* Do this as follows, write down a finite presentation

$$R^{\oplus m} \longrightarrow R^{\oplus n} \longrightarrow M \longrightarrow 0$$

Then apply the functor $\mathrm{Hom}_R(-, \omega_R)$ to get

$$0 \longrightarrow \mathrm{Hom}_R(M, \omega_R) \longrightarrow \omega_R^{\oplus n} \longrightarrow N \longrightarrow 0$$

where N is a submodule of $\omega_R^{\oplus m}$. In particular, $\omega_R^{\oplus n}$ satisfies $\mathbf{S}_2$ whereas N satisfies $\mathbf{S}_1$. Conclude from here that $\mathrm{Hom}_R(M, \omega_R)$ satisfies $\mathbf{S}_2$ (use the beahvior of depth on exact sequences).

Exercise 5.25. Let R be an $\mathbf{S}_1$ ring admitting a canonical module ω_R and M be an f.g. R-module. Prove that the canonical map

$$M \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(M, \omega_R), \omega_R)$$

is an isomorphism (resp. injective) if and only if M satisfies $\mathbf{S}_2$ (resp. $\mathbf{S}_1$). Conclude that:

(a) $R \longrightarrow \mathrm{Hom}_R(\omega_R, \omega_R)$ is an isomorphism if and only if R satisfies $\mathbf{S}_2$.
(b) If R satisfies $\mathbf{S}_2$ then $H^{\mathrm{ht}\,\mathfrak{p}}_{\mathfrak{p}}(\omega_R)$ is an injective hull at every point $\mathfrak{p} \in \mathrm{Spec}\, R$.

Hint: Let $(-)^{\omega} := \mathrm{Hom}_R(-, \omega_R)$ be the *Serre dual*. To show that $M \longrightarrow M^{\omega\omega}$ is an isomorphism if M satisfies $\mathbf{S}_2$ proceed as follows. First, let's look at the kernel. It has to vanish in codimension 0 and so generically. This means that it is supported in codimension ≥ 1. On the other hand, since it is a submodule of M; which is $\mathbf{S}_2$ by hypothesis, the kernel must be $\mathbf{S}_1$. This implies that the kernel is zero. Let's look next at the cokernel:

$$0 \longrightarrow M \longrightarrow M^{\omega\omega} \longrightarrow C \longrightarrow 0.$$

Observe that C is supported only in codimension ≥ 2. Suppose, for the sake of contradiction, that $C \neq 0$ and let $\mathfrak{p}$ be an associated prime of $\mathfrak{p}$, which must have height at least 2. In particular, $\operatorname{depth} C_{\mathfrak{p}} = 0$. Localizing the above sequence at $\mathfrak{p}$ yields

$$0 \longrightarrow M_{\mathfrak{p}} \longrightarrow M^{\omega\omega}_{\mathfrak{p}} \longrightarrow C_{\mathfrak{p}} \longrightarrow 0.$$

Use that both M and $M^{\omega\omega}$ satisfy $\mathbf{S}_2$ and the behavior of depth in exact sequences to obtain a contradiction.

Remark 5.18. Let R be an $\mathbf{S}_1$ ring admitting a canonical module ω_R. Because of the above, the natural transformation $M \longrightarrow M^{\omega\omega}$ on f.g. $\mathbf{S}_1$ modules is often referred to as *$\boldsymbol{S}_2$-ification*. It's often more useful when R itself satisfies $\mathbf{S}_2$. As we've observed before, and $\mathbf{S}_2$ module that vanishes in codimension ≤ 1 must be zero. More generally, we can say the following; which is one of the main reasons why $\mathbf{S}_2$ modules are awesome. The proof is left as an exercise.

Proposition 5.19. *Let R be an $\boldsymbol{S}_1$ ring admitting a canonical module ω_R and $\phi\colon M \longrightarrow N$ be a homomorphism of R-modules. The following statements hold:*

(a) Suppose that ϕ is injective in codimension 0. Then, $\phi^{\omega\omega}\colon M^{\omega\omega} \longrightarrow N^{\omega\omega}$ is injective. In particular, if M satisfies $\boldsymbol{S}_1$ then ϕ is injective.
(b) Suppose that ϕ is an isomorphism in codimension 1. Then $\phi^{\omega\omega}\colon M^{\omega\omega} \longrightarrow N^{\omega\omega}$ is an isomorphism. In particular, ϕ is an isomorphism if M and N are both $\boldsymbol{S}_2$.

Remark 5.20. In the setup of Proposition 5.19, I don't think that's true that if ϕ is surjective in codimension 1 and M and N are both $\mathbf{S}_2$ then ϕ is surjective. I'll try to add a counterexample at some point. But here's the thing. If ϕ isn't injective in codimension 1, then you can only say that it's image is $\mathbf{S}_1$ and that isn't enough to conclude. Or, more geometrically, what's happening is that the cokernel of ϕ is supported in codimension 2. Let $i\colon U \subset \operatorname{Spec} R$ be the open where $\operatorname{coker} \phi$ vanishes. The process of reflexification is the functor $i_* i^*$ (for those familiar with basic scheme theory). The restriction functor i^* is certainly exact and that's no issue. However, i_* isn't necessarily exact. It's left exact but not necessarily right exact. And so it can mess up with your cokernel. That's the issue really. We also see the following.

Proposition 5.21. *Let R be a ring and $\phi\colon M \longrightarrow N$ be a homorphism of R-modules. Then:*

(a) *If ϕ is injective in codimension 0 and M is $\boldsymbol{S}_1$ then ϕ is injective.*
(b) *If ϕ is an isomorphism in codimension 1 and both M and N are $\boldsymbol{S}_2$ then ϕ is an isomorphism.*

Definition 5.22. A ring R is said to be *normal* if and only if $R_{\mathfrak{p}}$ is an integrally closed domain for all $\mathfrak{p} \in \operatorname{Spec} R$.

Exercise 5.26. Show that a normal ring is a finite product of normal integral domains. Prove that an integral domain is normal if and only if it is integrally closed.

Theorem 5.23 (Serre's $\mathbf{R}_1 + \mathbf{S}_2$ criterion for normality)**.** *A ring is normal if and only if it satisfies $\boldsymbol{R}_1 + \boldsymbol{S}_2$.*

Corollary 5.24. *Regular rings are normal and the converse is true in dimension 1. Moreover, a normal ring of dimension ≤ 2 is Cohen–Macaulay.*

The next ingredient we need is *standard reflexification*.

Definition 5.25. We define the contravariant functor $(-)^\vee = \operatorname{Hom}_R(-, R)$ and consider the *reflexification* natural transformation

$$\alpha\colon \operatorname{id} \longrightarrow (-)^{\vee\vee}$$

given by

$$\alpha_M\colon M \xrightarrow{m \mapsto (\mu \mapsto \mu(m))} \operatorname{Hom}_R(\operatorname{Hom}_R(M, R), R)$$

We only consider this on finitely generated modules. We say that M is *reflexive* if and only if α_M is an isomorphism. In general, we referred to $M^{\vee\vee}$ as the *reflexification* of M when M satisfies $\mathbf{S}_1$.

Exercise 5.27. Prove that a flat (f.g.) module is reflexive.

Exercise 5.28. Let R be a ring that is $\mathbf{G}_1 + \mathbf{S}_2$ (*e.g.* normal). Prove that an f.g. R-module M is reflexive iff it is $\mathbf{S}_2$.

Exercise 5.29. Let M be a finitely generated module over an integral domain R. Prove that the following statements are equivalent:

(a) M is torsion-free.
(b) α_M is injective.
(c) M satisfies $\mathbf{S}_1$.
(d) M is a submodule of a (locally) free module (finitely generated).

Exercise 5.30. Let M be a finitely generated module over an integral domain R. Prove that M is reflexive if and only if it sits in a short exact sequence

$$0 \longrightarrow M \longrightarrow L \longrightarrow N \longrightarrow 0$$

where L is a finitely generated free module and N is torsion-free. Use this to conclude that $M^\vee$ is reflexive. Furthermore, prove that $\operatorname{Hom}_R(M, N)$ is reflexive if so is N.

The following result is a first main reason why normal rings are absolutely awesome and we want them around.

Corollary 5.26. *Let R be a normal integral domain and M be an f.g. R-module. Then, M is reflexive iff it satisfies $\boldsymbol{S}_2$.*

Remark 5.27. We're gonna use this as follows. Very often we'll have a morphism between certain reflexive modules such as suitable Hom modules. Then to show it is an isomorphism all we'll have to do is to show it in codimension ≤ 1 where it'll be often obvious. What a neat trick, innit?

It's time to talk about divisors. They're a fantastic bookkeeping tool (over normal integral domains).

Definition 5.28. Let R be a normal integral domain.

(a) A *prime divisor* on $\operatorname{Spec} R$ is a closed subset $P = V(\mathfrak{p})$ where $\mathfrak{p}$ is a prime ideal of height 1.[30]

(b) The *group of (Weil) divisors* $\operatorname{Div} R$ is the free abelian group generated by prime divisors.

(c) A *(Weil) divisor* on $\operatorname{Spec} R$ is an element $D \in \operatorname{Div} R$. That is, a divisor D is nothing but a formal finite sum
$$D = \sum_P a_P P$$
where the P's are prime divisors and $a_P \in \mathbb{Z}$ (all but finitely many of these coefficient sare zero). We may sometimes write
$$a_P = \operatorname{ord}_P D = \operatorname{val}_P D$$
as the *order (of vanishing)* of D at P.

(d) A divisor D is said to be *effective* if $\operatorname{val}_P D \geq 0$ for all P. We write $D \geq 0$ say that D is effective.

(e) We may also take a ring A (but essentially we only care about $A = \mathbb{Q}$ or $\mathbb{Z}_{(p)}$) and define *A-divisors* as elements of $\operatorname{Div}_A R \coloneqq A \otimes_{\mathbb{Z}} \operatorname{Div} R$.[31]

Let R be a normal integral domain with field of fractions K. Since R is normal, each local ring $R_\mathfrak{p}$ is a DVR for all $\mathfrak{p} \in \operatorname{Spec} R$ of height 1.[32] And so, it comes equipped with a discrete valuation
$$\operatorname{val}_P \colon K^\times \longrightarrow \mathbb{Z}$$
where P is the prime divisor corresponding to $\mathfrak{p}$. So we can compute the order of vanishing (or pole) of every $f \in K^\times$ at every single prime divisor. Moreover,

[30] A prime divisor is nothing but a prime ideal of height 1. But it's convenient (to say the least) to think of them more geometrically.

[31] Which is the same thing as before but the coefficients are taken in A.

[32] This because a local ring of dimension 1 is the same thing as a DVR.

Exercise 5.31. With notation as above, prove the following:

(a) Let $f \in K^\times$. The integer $\mathrm{val}_P f$ is zero for all but finitely many prime divisors P.
(b) This defines a group homomorphism
$$\mathrm{div}\colon K^\times \longrightarrow \operatorname{Div} R, \quad f \mapsto \sum_P (\mathrm{val}_P f) P$$
(c) with kernel $\ker \mathrm{div} = R^\times$—the group of units of R.

Definition 5.29 (Principal divisors and the divisor class group)**.** With notation as above, the image of div is the group of *principal divisors*. The cokernel of div is defined as the *divisor class group* of R:
$$\operatorname{Cl} R \coloneqq \operatorname{coker} \mathrm{div} = \operatorname{Div} R / \operatorname{div}(K^\times) = \operatorname{Div} R / \sim$$
where two divisors $D_1, D_2 \in \operatorname{Div} R$ are *linearly equivalent* $D_1 \sim D_2$ iff $D_1 - D_2 = \operatorname{div} f$ for some $f \in K^\times$. Likewise, if A is a $\mathbb{Z}$-algebra (aka ring), we say that $D_1 \sim_A D_2$ for $D_1, D_2 \in \operatorname{Div}_A R$ if $D_1 = D_2$ in $A \otimes_{\mathbb{Z}} \operatorname{Cl} R = \operatorname{coker}(A \otimes \mathrm{div})$.

Exercise 5.32. Let $W \subset \mathbb{Z}$ be a multiplicative subset and $D_1, D_2 \in \operatorname{Div}_A R$. Prove that $D_1 \sim_{W^{-1}\mathbb{Z}} D_2$ iff there exists $n \in W$ such that $nD_1 \sim nD_2$.

Exercise 5.33. With notation as above, show that R is an UFD iff $\operatorname{Cl} R = 0$. If you feel like so, prove that
$$\operatorname{Cl}\left(\Bbbk[x_0, x_1, \dots, x_d]/(x_0^2 + x_1^2 + \cdots + x_d^2)\right) = \begin{cases} \mathbb{Z}/2 & \text{if } d = 2, \\ \mathbb{Z} & \text{if } d = 3, \\ 0 & \text{if } d \geq 4. \end{cases}$$
where $\operatorname{char} \Bbbk \neq 2$ and $d \geq 2$. The cases $d = 2, 3$ should be quite doable. The case $d \geq 4$ might be tough, but I believe in you.

Definition 5.30 (Divisorial/fractional module of a divisor)**.** Let R be a normal integral domain and $D \in \operatorname{Div} R$. Then we define the R-submodule
$$R(D) \coloneqq \{f \in K^\times \mid \operatorname{div} f + D \geq 0\} \cup \{0\} \subset K.$$

Exercise 5.34. Show that we have a bijection
$$(R(D) \setminus 0)/R^\times \xrightarrow{f \mapsto \operatorname{div} f + D} \{E \in \operatorname{Div} R \mid 0 \leq E \sim D\} =: |D|.$$
NB. The latter set is often referred in classical terms to a as the (full) *linear system* of D.

Exercise 5.35. With notation as in Definition 5.30, show the following.

(a) $R(D)$ has rank 1.
(b) The map
$$R(-D) \xrightarrow{f \mapsto \cdot f : (g \mapsto fg)} R(D)^\vee = \operatorname{Hom}_R(R(D), R)$$
is an isomorphism. Conclude that $R(D)$ is reflexive.
(c) A divisor D is effective iff $R(-D) \subset R$.
(d) For example, if $P = V(\mathfrak{p})$ is a prime divisor then $R(-P) = \mathfrak{p}$.
(e) If $f \in K^\times$ then $R(\operatorname{div} f) = R \cdot f^{-1} \cong R$.
(f) Conversely, if $R(D) \cong R$ then D is principal.
(g) More generally, $R(D_1) \cong R(D_1)$ iff $D_1 \sim D_2$.

Definition 5.31 (Cartier divisor)**.** We say that a divisor D is *Cartier* if $R(D)$ is flat, *i.e.* locally free of rank 1, *i.e.* invertible.

Since principal divisors are Cartier, the following definition makes sense.

Definition 5.32 (Picard group)**.** The *Picard group* of R is defined as the quotient of Cartier divisors modulo principal divisors. It is denoted by $\operatorname{Pic} R$.

Remark 5.33. The Picard group $\operatorname{Pic} R$ sits inside $\operatorname{Cl} R$ as the subgroup given by divisor classes of Cartier divisors.

Proposition 5.34. *A rank-1 reflexive R-submodule of K is equal to $R(D)$ for a uniquely determined divisor $D \in \operatorname{Div} R$. More precisely, let $\mathscr{I} \subset K$ be a reflexive R-submodule of rank 1. Consider the divisor*

$$D_{\mathscr{I}} := \sum_P a_P P$$

where

$$a_P := \operatorname{val}_P f$$

where $f \in \mathscr{I}$ is any generator of $\mathscr{I}_P \cong R_P$.[33] *Then,*

$$R(D_{\mathscr{I}}) = \mathscr{I} \text{ and } D_{R(D)} = D.$$

Proof. The point is that the canonical inclusion $\mathscr{I} \subset R(D_{\mathscr{I}})$ has to be an equality because it is an equality at every height-1 prime ideal and the modules in question are $\mathbf{S}_2$. □

More generally,

Proposition 5.35. *Let $\mathscr{I}$ be a rank 1 reflexive R-module. Then $\mathscr{I}$ can be embedded as a sumbodule of K and so it is isomorphic to $R(D)$ for some divisor D. Moreover, such divisor is unique up to linear equivalence. Further, $\mathscr{I}$ is invertible iff such divisor is Cartier.*

Proof. Since $\mathscr{I}$ is torsion-free, the canonical map $\mathscr{I} \longrightarrow K \otimes_R \mathscr{I}$ is injective. Since $\mathscr{I}$ has rank, $K \otimes_R \mathscr{I} \cong K$. In other words, $\mathscr{I}$ is a submodule of K up to the choice of a generic generator. □

Remark 5.36. The point is that reflexive R-submodules of rank 1 are nice conceptual objects we want to play with. But it's hard to play/compute with them. In contrast, divisors are good for the computations and to do things explicitly by simple arithmetic over $\mathbb{Z}$. The following illustrates this.

Exercise 5.36. Show that the following canonical maps are isomorphisms

(a)
$$(R(D_1) \otimes_R R(D_1))^{\vee\vee} \xrightarrow{f_1 \otimes f_2 \mapsto f_1 f_2} R(D_1 + D_2)$$
Show that if at least one D_1 or D_2 is Cartier, there's no need for the reflexification.

(b)
$$R(D_2 - D_1) \xrightarrow{f \mapsto \cdot f} \operatorname{Hom}_R\big(R(D_1), R(D_2)\big).$$

Remark 5.37. The isomorphism classes of rank 1 reflexive (resp. invertible) modules form a group under reflexified tensor product. We can conclude that these happen to be respectively isomorphic to the divisor class group and the Picard group.

[33]This last isomorphism relies on R_P being a PID and the structure theorem of f.g. modules over those. Since $\mathscr{I} \subset K$, it is torsion free and since it has rank 1 then $\mathscr{I}_P$ gotta be isomorphic to R_P.

5.6. **The canonical divisor.** We can finally define the canonical class on a normal integral domain. Let R be a ring admitting a canonical module ω_R. Recall that this meant that R is a finite equidimensional algebra over a Gorenstein ring of finite dimension. And these include F-finite rings (thanks to Gabber), complete rings (thanks to Cohen), and rings essentially of finite type over a field (thanks to Noether). However, ω_R is a reflexive module of rank 1 !!! Then we can write

$$\omega_R = R(K_R)$$

for some divisor K_R on $\operatorname{Spec} R$. Moreover, any such divisor is called *a canonical divisor* and is unique up to linear equivalence. So *the divisor class of* K_R, *i.e. the canonical class*, is unique.

Exercise 5.37. With notation as above, show that R is Gorenstein iff it is Cohen–Macaulay and K_R is a Cartier divisor.

However, technically speaking, a canonical divisor K_R depends on our choice of a canonical module ω_R, which is only unique up to tensoring by invertible modules. This means that K_R is only unique up to adding Cartier divisors. In the case R is further (essentially) of finite type over a field $\Bbbk$ (and more generally kählerian I believe). There's a more canonical way to define ω_R and so K_R. Namely, we may define the module of kähler differentials $\Omega_{R/\Bbbk}$, which is free of rank $\dim R$in codimension 1 and so over the regular locus of R. Then, one may consider $\tilde{\omega}_R$ its determinant over the regular locus, which is invertible over there. Then one may define ω_R as the unique reflexive R-module (of rank necessarily 1) that coincides with $\tilde{\omega}_R$ over the regular locus. The corresponding canonical divisor(s) is what people call canonical divisor in algebraic geometry, at least when dealing with normal varieties.

Definition 5.38 ($\mathbb{Q}$-Gorenstein rings)**.** With notation as above, R is said to be $\mathbb{Q}$-*Gorenstein* if K_R is torsion in $\operatorname{Cl} R/\operatorname{Pic} R$. It's order is referred to as the *Gorenstein index*.[34]

5.7. **Pullback of divisors and Schwede's correspondence.** Let $R \to S$ be a finite extension of normal integral domains. Let $f\colon \operatorname{Spec} S \to \operatorname{Spec} R$ be the corresponding morphism on spectra and L/K be the corresponding extension of fields of fractions. Then, if P is a prime divisor on R we can write

$$R_P \to S_P$$

for the corresponding localization. Then, S_P is a semi-local Dedekind domain and so a PID. Further,

$$S_P = S_{Q_1} \cap \cdots \cap S_{Q_k}$$

where $Q_1, \dots, Q_k$ is the list of (finitely many) prime divisors on $\operatorname{Spec} S$ lying over P. In particular, we obtain the extensions of DVRs:

$$R_P \subset S_{Q_i}, \quad i = 1, \dots, k$$

with corresponding *ramification index* $0 \neq e_i \in \mathbb{N}$. This is the info we need to pull back divisors (along finite extensions). We define:

$$f^*P \coloneqq e_1 Q_1 + \cdots e_k Q_k$$

for a prime divisor P on $\operatorname{Spec} R$ and we extend it to a homomorphism

$$f^*\colon \operatorname{Div}_A R \to \operatorname{Div}_A S$$

[34]However, having Gorenstein index 1 implies Gorenstein only in the Cohen–Macaulay case.

by linearity. In other words, all the pullback does is to keep a record of the prime divisors lying over a given divisor and the corresponding ramification indexes. Moreover,

Exercise 5.38. Show that $f^* \operatorname{div}_R r = \operatorname{div}_S r$ for all $r \in K^\times$. Conclude that if $D_1 \sim D_2$ then $f^*D_1 \sim f^*D_2$ and so that we get a homomorphism

$$f^*\colon \operatorname{Cl} R \longrightarrow \operatorname{Cl} S$$

Exercise 5.39. Suppose that $R \subset S$ is the Frobenius $F^e\colon R \longrightarrow R$. Prove that

$$F^{e*}D = qD$$

for all divisors D on $\operatorname{Spec} R$.

Exercise 5.40. Show that the canononical map

$$f^*R(D) \coloneqq S \otimes_R R(D) \longrightarrow S(f^*D)$$

is the reflexification. That is, show that $f^*R(D)$ is $\mathbf{S}_1$ and that this map is an isomorphism up to reflexification.

Exercise 5.41 (Projection formula)**.** Let $R \longrightarrow S$ be a homomorphsim of rings. Let N be an f.g. S-module and M be an f.g. R-modules. Prove that the canonical map

$$f_*N \otimes_R M \longrightarrow f_*(N \otimes_S f^*M)$$

is an isomorphism if M is flat. Suppose further $R \longrightarrow S$ is a finite extension of normal integral domains and that $N = S(E)$ and $M = R(D)$ are reflexive modules of rank 1. Show that

$$f_*S(E) \otimes_R R(D) \longrightarrow f_*\big(S(E + f^*D)\big)$$

is a reflexification.

Exercise 5.42. Conclude that there's a canonical isomorphism

$$\operatorname{Hom}_R(f_*S(E), R(D)) \xleftrightarrow[(\psi\otimes R(D))^{\vee\vee}\leftarrow\!\shortmid\psi]{\phi\mapsto(\phi\otimes R(-D))^{\vee\vee}} \operatorname{Hom}_R(f_*S(E - f^*D), R).$$

This isomorphism can also be thought of as follows. Note that, generically, both sides of the isomorphism are generically the same thing, namely $\operatorname{Hom}_K(L, K)$. In fact, each of them can be thought of as L-submodules of $\operatorname{Hom}_K(L, K)$. Indeed, $\operatorname{Hom}_R(f_*S(E), R(D))$ is the submodule of those maps $L \longrightarrow K$ that send $S(E)$ into $R(D)$. Likewise, $\operatorname{Hom}_R(f_*S(E - f^*D), R)$ are those maps in $\operatorname{Hom}_K(L, K)$ sending $S(E - f^*D)$ into R. The isomorphism above says that we can go to another another molding domains and co-domains appropriately.

Suppose now that $\omega_R = R(K_R)$ and $\omega_S = S(K_S)$ are canonical modules over R and S; respectively. Moreover, let's fix an isomorphism $\omega_S \cong f^!\omega_R$ providing us with a trace

$$\operatorname{Tr} = \operatorname{Tr}_{\omega_R}\colon f_*\omega_S \longrightarrow \omega_R.$$

In particular, we have an isomorphism of S-modules

$$S \xrightarrow{1\mapsto \mathrm{id}} \operatorname{Hom}_S(\omega_S, \omega_S = f^!\omega_R) \xrightarrow{\mathrm{id}\mapsto \operatorname{Tr}_{\omega_R}} \operatorname{Hom}_R(f_*\omega_S, \omega_R)$$

as S is $\mathbf{S}_2$. Then, we have the following isomorphism of S-modules:

$$\begin{aligned}\operatorname{Hom}_R(f_*S(E), R(D)) &\cong \operatorname{Hom}_R(f_*S(E - f^*D + f^*K_R), R(K_R)) \\ &\cong \operatorname{Hom}_S(S(E - f^*D + f^*K_R), S(K_S)) \\ &\cong S(K_{S/R} - (E - f^*D)).\end{aligned}$$

where $K_{S/R} \coloneqq K_f \coloneqq K_S - f^*K_R$. This isomorphism can be made very explicit. First, using Exercise 5.42, we may think of the trace $\operatorname{Tr}_{\omega_R} \colon f_*S(K_S) \longrightarrow R(K_R)$ as a map

$$\operatorname{Tr}_R \coloneqq (\operatorname{Tr} \otimes R(-K_R))^{\vee\vee} \colon F^e_* S(K_S - f^*K_R) \longrightarrow R$$

That is,

$$\operatorname{Tr}_R \colon F^e_* S(K_{S/R}) \longrightarrow R$$

Remark 5.39. In fact, if we look at Tr generically, it is just a map $\operatorname{Tr}_K \colon L \longrightarrow K$. This says that $S(K_{S/R}) \subset L$ is the largest reflexive S-submodule of rank 1 such that is sent into R by Tr_K.

In the same spirit, we can also think of it as a map:

$$\operatorname{Tr}_{R(D)} \colon F^e_* S(K_{S/R} + f^*D) \longrightarrow R(D)$$

Now, if we take

$$s \in S(K_{S/R} - (E - f^*D))$$

that is

$$\operatorname{div} s + K_{S/R} + f^*D \geq E$$

then it induces and S-linear map

$$S(E) \xrightarrow{\cdot s} S(K_{S/R} + f^*D)$$

In fact, any such map is obtained in this way. In particular, we can campose with $\operatorname{Tr}_{R(D)}$ to obtain

$$\operatorname{Tr}_{R(D)} s \colon S(E) \xrightarrow{\cdot s} S(K_{S/R} + f^*D) \xrightarrow{\operatorname{Tr}_{R(D)}} R(D)$$

Summing up, the following map is an isomorphism of S-modules

$$S(K_{S/R} + f^*D - E) \xrightarrow{s \mapsto \operatorname{Tr}_{R(D)} s} \operatorname{Hom}_R(f_*S(E), R(D))$$

Exercise 5.43. Show that if $E = 0$; so that on the right-hand side we have $f^!R(D)$, then this isomorphism identifies the map $\operatorname{Tr}_{R(D)}$ with the trace $f_*f^!R(D) \longrightarrow R(D)$ of $R(D)$. In particular, this justifies the above notation. This is why we sometimes use the notation

$$f^!D \coloneqq f^*D + K_{S/R}$$

so that we may write

$$\operatorname{Tr}_{R(D)} \colon f_*R(f^!D) \longrightarrow R(D).$$

Now, using Exercise 5.34, we obtain a bijection

$$(\operatorname{Hom}_R(f_*S(E), R(D)) \setminus 0)/S^\times \xrightarrow{\phi \mapsto D_\phi} |K_{S/R} + f^*D - E|$$

In other words,

- to every nonzero S-linear map $\phi \colon S(E) \longrightarrow R(D)$ there corresponds an effective divisor D_ϕ over S that is linearly equivalent to $K_{S/R} + f^*D - E$,
- two maps have the same divisor iff they differ to one another by premultiplication by units of S, and

- every divisor in $|K_{S/R} + f^*D - E|$ is obtained in this way.

Exercise 5.44. Prove that $\phi \in \mathrm{Hom}_R(f_*S(E), R(D))$ is a free generator as an S-module iff $D_\phi = 0$.

This correspondence is functorial in the following sense:

Exercise 5.45 (Functoriality)**.** Let $R \subset S \subset T$ be a tower of finite extensions of normal integral domains. Let D, E, and G be divisors over R, S, and T; respectively. Let's take a couple of nonzero maps

$$T(G) \xrightarrow{\psi} S(E) \xrightarrow{\phi} R(D).$$

Prove that

$$D_{\psi\circ\phi} = D_\psi + g^* D_\phi$$

where $g\colon \operatorname{Spec} T \longrightarrow \operatorname{Spec} S$ is the corresponding map on spectra.

Definition 5.40. We set

$$\omega_f \coloneqq \omega_{S/R} \coloneqq f^! R = \mathrm{Hom}_R(S, R) \xleftarrow{\cong} S(K_{S/R})$$

and refer to it as the *relative canoninical module* of S/R or simple as the *canonical module* of f. Likewise, we refer to $K_f = K_{S/R} = K_S - f^*K_R$ as the *relative canonical divisor.*

Remark 5.41*.* Technically speaking, Ram is only linearly equivalent to $K_{S/R}$, so the above is an abuse of notation.

Finally, we're ready to apply this to the Frobenius map $F^e\colon R \longrightarrow R$ for a ring R admitting a canonical module such that

$$F^! \omega_R \cong \omega_R.$$

The first thing to observe is that

$$\mathrm{Ram}_{F^e} = (1-q)K_R$$

And recall that the Cartier operator $\kappa_R^e : F^e_* R(K_R) \longrightarrow R(K_R)$ is (by definition) our trace. In particular, we'll write

$$\mathrm{Tr}^e_D = \kappa^e_{R(-K_R+D)} \colon F^e_* R((1-q)K_R + qD) \longrightarrow R(D)$$

for $(\kappa^e_R \otimes R(-K_R + D))^{\vee\vee}$. All the above tells us that

$$F^e_* R((1-q)K_R + qD - E) \xrightarrow{s \mapsto \mathrm{Tr}^e_D s} \mathrm{Hom}_R(F^e_* R(E), R(D))$$

is an isomorphism of F^e_*R-modules. Moreover, this induces a bijection

$$(\mathrm{Hom}_R(F^e_* R(E), R(D)) \setminus 0)/(F^e_* R)^\times \xrightarrow{\phi \mapsto D_\phi} |(1-q)K_R + qD - E|$$

which we may spell out as follows:

- to every nonzero R-linear map $\phi\colon F^e_* R(E) \longrightarrow R(D)$ there corresponds an effective divisor D_ϕ over R that is linearly equivalent to $(1-q)K_R + qD - E$,
- two maps have the same divisor iff they differ to one another by premultiplication by units of R, and
- every divisor in $|(1-q)K_R + qD - E|$ is obtained in this way.

Now, setting $D = E$ and defining the effective $\mathbb{Z}_{(p)}$-divisor

$$\Delta_\phi := \frac{1}{q-1} D_\phi$$

we can say that: to every nonzero R-linear map $\phi\colon F^e_* R(E) \longrightarrow R(E)$ there corresponds an effective $\mathbb{Z}_{(p)}$-divisor Δ_ϕ over R such that

$$K_R + \Delta_\phi \sim_{\mathbb{Z}_{(p)}} E.$$

Definition 5.42. We say that two maps $0 \neq \phi\colon F^e_* R(E) \longrightarrow R(E)$ and $0 \neq \psi\colon F^d_* R(E) \longrightarrow R(E)$ are Δ-*equivalent* and we write $\phi \sim_\Delta \psi$ if e and d have a common multiple n such that $\phi^{n/e} = \psi^{n/d} \cdot u$ for some unit $u \in R$. Recall that

$$\phi^k := \phi \circ F^e_* \phi \circ F^{2e}_* \phi \circ \cdots \circ F^{(k-1)e}_* \phi\colon F^{ek}_* R(E) \longrightarrow R(E).$$

Exercise 5.46. Verify that Δ-equivalence is an equivalence relation on the set

$$\bigsqcup_{e \in \mathbb{N}} \operatorname{Hom}_R(F^e_* R(E), R(E)) \setminus 0.$$

Exercise 5.47. Show that $\phi \sim_\Delta \psi$ iff $\Delta_\phi = \Delta_\psi$.

Exercise 5.48. Prove that

$$\mathbb{Z}_{(p)} = \operatorname{colim}_e \mathbb{Z}_{q-1},$$

where $\mathbb{Z}_{q-1} = \mathbb{Z}[1/(q-1)] \subset \mathbb{Q}$.

You have now all you need to conclude the following, which is the bridge between F-singularities and Mori's singularities in birational geometry—the singularities of the Minimal Model Program.

Theorem 5.43 (Schwede's correspondence I)**.** *With notation as above, to every nonzero R-linear map $\phi\colon F^e_* R(E) \longrightarrow R(E)$ there corresponds an effective $\mathbb{Z}_{(p)}$-divisor Δ_ϕ over R such that*

$$K_R + \Delta_\phi \sim_{\mathbb{Z}_{(p)}} E.$$

Moreover, any such divisor is obtained in this way and two maps define the same $\mathbb{Z}_{(p)}$-divisor if and only if a power of them differ by premultiplication by units of R. More succinctly, there's a bijection for a fixed divisor E over R:

$$\left(\bigcup_{e \in \mathbb{N}} \operatorname{Hom}_R(F^e_* R(E), R(E)) \setminus 0 \right) \Big/ \sim_\Delta \xrightarrow{\phi \mapsto \Delta_\phi} \{\textit{effective } \mathbb{Z}_{(p)}\textit{-divisors s.t. } K_R + \Delta \sim_{\mathbb{Z}_{(p)}} E\}$$

Exercise 5.49. Prove that $\phi \in \operatorname{Hom}_R(F^e_* R(E), R)$ is a free generator (as an $F^e_* R$-module) iff $\Delta_\phi = 0$.

Question 5.44. What happens if one p-adically completes? That is, what does an "effective" $\mathbb{Z}_p$-divisor[35] Δ such that $K_R + \Delta \sim_{\mathbb{Z}_p} E$ correspond to? Likewise, what does an effective $\mathbb{Z}[1/p]$-divisor Δ such that $K_R + \Delta \sim_{\mathbb{Z}[1/p]} E$ correspond to?

[35]Here $\mathbb{Z}_p := \hat{\mathbb{Z}}_{(p)}$ denotes the ring of p-adic integers.

Likewise, we could have set $D = 0$. In that case, to every nonzero R-linear map $\phi\colon F^e_* R(E) \longrightarrow R$ there corresponds an effective $\mathbb{Z}_{(p)}$-divisor Δ_ϕ over R such that

$$(q-1)(K_R + \Delta_\phi) \sim E$$

In particular, if E is Cartier then $K_R + \Delta_\phi$ is $\mathbb{Z}_{(p)}$-Cartier, *i.e.* there is $0 \neq n \in \mathbb{N}$ prime-to-p such that $n(K_R + \Delta)$ is a Cartier $\mathbb{Z}$-divisor.

Definition 5.45. We refer to an effective divisor $\mathbb{Z}_{(p)}$-divisor Δ over R such that $K_R + \Delta$ is $\mathbb{Z}_{(p)}$-Cartier as a *tame boundary*.[36] We say that two maps $0 \neq \phi\colon F^e_* R(E) \longrightarrow R$ and $0 \neq \psi\colon F^d_* R(D) \longrightarrow R$ with E and D Cartier divisors are *boundary equivalent* if e and d have a common multiple n such that

$$\frac{p^n - 1}{p^e - 1} E \sim \frac{p^n - 1}{p^d - 1} D$$

and the induced diagram

$$\begin{tikzcd} F^n_* R\left(\frac{p^n-1}{p^e-1}E\right) \arrow[rr, leftrightarrow, "\cong"] \arrow[dr, "\phi^{n/e}"'] & & F^n_* R\left(\frac{p^n-1}{p^d-1}D\right) \arrow[dl, "\psi^{n/d}"] \\ & R & \end{tikzcd}$$

commutes, where powers are defined inductively as

$$\phi^k \coloneqq \phi \circ F^e_*(\phi^{k-1} \otimes R(E))^{\vee\vee}.$$

In that case, we write $\phi \sim_B \psi$.

Exercise 5.50. With notation as in Definition 5.45,show that

$$\phi \sim_B \psi \iff \Delta_\phi = \Delta_\psi$$

Theorem 5.46 (Schwede's correspondence II)**.** *With notaion as above, there's a bijection*

$$\left(\bigcup_{\substack{E\colon \textit{Cartier divisor} \\ e \in \mathbb{N}}} \operatorname{Hom}_R(F^e_* R(E), R) \setminus 0 \right) \Big/ \sim_B \xrightarrow{\phi \mapsto \Delta_\phi} \{\textit{Tame Boundaries}\}$$

Exercise 5.51. Let R be $\mathbb{Q}$-Gorenstein with a prime-to-p Gorenstein index n, *i.e.* R is $\mathbb{Z}_{(p)}$-Gorenstein. Prove that there is $e \in \mathbb{N}$ such that $\operatorname{Hom}_R(F^e_* R, R)$ is an invertible $F^e_* R$-module. Can you compute e explicitly in terms of n? If $K_R \in \operatorname{Cl} R$ is further torsion, show that there is $e \in \mathbb{N}$ such that $\operatorname{Hom}_R(F^e_* R, R)$ is further free (of rank 1) as an $F^e_* R$-module.

And to close this chapter with flourish, we have the following:

Theorem 5.47. *Let R be a normal integral domain with Cartier operators $\kappa^e_R : F^e_* R(K_R) \longrightarrow R(K_R)$. Then, R is F-pure iff κ^e_R splits.*[37]

[36]With $\mathbb{Q}$ in place of $\mathbb{Z}_{(p)}$, one talks about plain *boundaries*.

[37]Whereas, if R is Cohen–Macaulay, R is F-injective if and only if κ^e_R is surjective. Of course, both conditions splitness and surjectivity are the same if the target $R(K_R)$ is projective, *i.e.* K_R is Cartier, *i.e.* R is further Gorenstein.

Proof. Note that $\kappa_R^e : F_*^e R(K_R) \longrightarrow R(K_R)$ is split if and only if so is

$$\mathrm{Tr}_D^e = \kappa_{R(-K_R+D)}^e \colon F_*^e R((1-q)K_R + qD) \longrightarrow R(D)$$

for any/all divisors D. The case $D = 0$ is imply the case of the trace

$$\mathrm{Hom}_R(F_*^e R, R) \xrightarrow{\phi \mapsto \phi(F_*^e 1)} R$$

whose surjectivity/splitness is the same as the F-purity of R (as R is F-finite). $\square$

6. Cartier Modules, Test Modules, and Centers of F-purity

Let R be an F-finite $\mathbb{F}_p$-algebra. Let's start off rather abstractly:

Definition 6.1 (Cartier algebra)**.** A *Cartier algebra* over R is an $\mathbb{N}$-graded non-necessarily commutative ring

$$\mathscr{C} \coloneqq \bigoplus_{e \in \mathbb{N}} \mathscr{C}_e$$

where $\mathscr{C}_0 = R$ and such that the induced R-bimodule structure on $\mathscr{C}_e$ is such that

$$r\kappa = \kappa r^q$$

for all $r \in R$ and $\kappa \in \mathscr{C}_e$. I think we may want to take as part of the definition that each $\mathscr{C}_e$ is finitely generated as both left and right R-module. A homomorphism of Cartier algebras is defined in the expected ways: it is a homomorphism of rings that respect the R-bimodule structure (in particular, it's the identity on $\mathscr{C}_0 = R$). Moreover, we set:

$$\mathscr{C}_+ \coloneqq \bigoplus_{0 \neq e \in \mathbb{N}} \mathscr{C}_e$$

Example 6.2 (Free Cartier algebra)**.** We may take $\mathscr{C} = R\{\kappa\}/(r\kappa = \kappa r^p)$. That is $\mathscr{C}$ is the quotient of the non-commutative polynomial algebra $R\{\kappa\}$ by the ideal $(r\kappa - \kappa r^p \mid r \in R)$. In this case,

$$\mathscr{C}_e = \kappa^e \cdot R.$$

We may have also taken $\mathscr{C} = R\{\kappa\}/(r\kappa = \kappa r^q)$. In that case, $\mathscr{C}_d = 0$ is $d \not\equiv 0 \bmod e$ and

$$\mathscr{C}_{ke} = \kappa^k \cdot R.$$

Moreover, we could have also set

$$R\{\kappa_1, \ldots, \kappa_n\}/(r\kappa_1 = \kappa_1 r^{q_1}, \ldots, r\kappa_n = \kappa_n r^{q_n})$$

and have as much fun as you want.

Definition 6.3 (Finite generation)**.** We say that a Cartier algebra $\mathscr{C}$ is *finitely generated* is there is a surjective homomorphism of Cartier algebras

$$R\{\kappa_1, \ldots, \kappa_n\}/(r\kappa_1 = \kappa_1 r^{q_1}, \ldots, r\kappa_n = \kappa_n r^{q_n}) \twoheadrightarrow \mathscr{C}$$

for some $e_1, \ldots, e_n \in \mathbb{N}$.

Example 6.4 (Full Cartier algebras)**.** The *full Cartier algebra* of an f.g. R-module M is

$$\mathscr{C}_M \coloneqq \bigoplus_{e \in \mathbb{N}} \mathrm{Hom}_R(F_*^e M, M)$$

where

$$\phi \cdot \psi \coloneqq \phi \circ F_*^e \psi \in \mathscr{C}_{M, e+d}$$

for all $\phi \in \mathscr{C}_{M,e}$ and $\psi \in \mathscr{C}_{M,d}$ and all $e, d \in \mathbb{N}$. For instance, if R is further a normal integral domain (admitting a canonical divisor) and D is a divisor over it:

$$\mathscr{C}_{R(D)} \cong \bigoplus_{e \in \mathbb{N}} R((1-q)(K_R - D))$$

where multiplication is given by

$$f \cdot g = f^{q'} g \in R((1-qq')(K_R - D))$$

for all $f \in R((1-q)(K_R - D))$, $g \in R((1-q')(K_R - D))$, and all $e, e' \in \mathbb{N}$.

Further, if $\phi_1, \ldots, \phi_n \in \mathscr{C}_M$ are homogeneous elements, say $\phi_i \in \mathscr{C}_{M,e_i}$, we define the Cartier subalgebra

$$\mathscr{C}_M^{\phi_1,\ldots,\phi_n} \subset \mathscr{C}_M$$

generated by them as the image of

$$R\{\kappa_1, \ldots, \kappa_n\}/(r\kappa_1 = \kappa_1 r^{q_1}, \ldots, r\kappa_n = \kappa_n r^{q_n}) \xrightarrow{\kappa_i \mapsto \phi_i} \mathscr{C}_M.$$

Exercise 6.1. Prove that the full Cartier algebra $\mathscr{C}_R$ of a $\mathbb{Z}_{(p)}$-Gorenstein local ring is finitely generated.

Example 6.5. Let M be an f.g. R-module, $\mathfrak{a} \subset R$ an ideal, and t a positive real number. Then, one defines the Cartier subalgebra $\mathscr{C}_M^{\mathfrak{a}^t} \subset \mathscr{C}_M$ as follows:

$$\mathscr{C}_M^{\mathfrak{a}^t} \coloneqq \bigoplus_{e \in \mathbb{N}} \mathrm{Hom}_R(F_*^e M, M)\mathfrak{a}^{\lceil t(q-1) \rceil}$$

Verify that this is indeed a Cartier subalgebra.

Example 6.6. Let R be a normal integral domain, $D \in \mathrm{Div}\, R$, and $0 \leq \Delta \in \mathrm{Div}_{\mathbb{R}}\, R$. Recall that

$$F_*^e R((1-q)(K_R - D)) \xrightarrow{s \mapsto \mathrm{Tr}_D^e\, s} \mathrm{Hom}_R(F_*^e R(D), R(D)) = \mathscr{C}_{e,R(D)}$$

is an isomorphism. From this, we obtained that to a map $0 \neq \phi \in \mathscr{C}_{e,R(D)}$ there corresponds $0 \neq \Delta_\phi \in \mathrm{Div}_{\mathbb{Z}_{(p)}} R$ such that

$$K_R + \Delta_\phi \sim_{\mathbb{Z}_{(p)}} D.$$

We define

$$\mathscr{C}_{e,R(D)}^{\Delta} \coloneqq \{\phi \in \mathscr{C}_{e,R(D)} \mid \Delta_\phi \geq \Delta\}.$$

Observe that this corresponds to the image of

$$F_*^e R((1-q)(K_R - D) - \lceil (q-1)\Delta \rceil) \xrightarrow{s \mapsto \mathrm{Tr}_D^e\, s} \mathrm{Hom}_R(F_*^e R(D + \lceil (q-1)\Delta \rceil), R(D))$$

inside $\mathscr{C}_{e,R(D)}$. In particular,

$$\mathscr{C}_{R(D)}^{\Delta} \cong \bigoplus_{e \in \mathbb{N}} R((1-q)(K_R - D) - \lceil (q-1)\Delta \rceil)$$

It's also customary to consider Cartier algebras $\mathscr{C}_{R(D)}^{\Delta,\mathfrak{a}^t}$. We then have

$$\mathscr{C}_{R(D)}^{\Delta,\mathfrak{a}^t} \cong \bigoplus_{e \in \mathbb{N}} \mathfrak{a}^{\lceil t(q-1) \rceil} R((1-q)(K_R - D) - \lceil (q-1)\Delta \rceil)$$

Exercise 6.2. Suppose that $0 \leq \Delta \in \mathrm{Div}_{\mathbb{Z}_{(p)}} R$ is such that $K_R + \Delta \sim_{\mathbb{Z}_{(p)}} D$. Describe $\mathscr{C}_{R(D)}^{\Delta}$.

Exercise 6.3. Let R be a normal integral domain, $0 \neq f \in R$, and $0 \leq t \in \mathbb{R}$. Set $\mathfrak{a} := (f)$ and $\Delta := t \operatorname{div} f$. How does $\mathscr{C}_R^{\mathfrak{a}^t}$ and $\mathscr{C}_R^{\Delta}$ compare to one another?

Example 6.7. Let $\mathfrak{a} \subset R$ be an ideal. We define the Cartier algebra $\mathscr{C}_R^{[\mathfrak{a}]}$ of $\mathfrak{a}$-*maps* as

$$\mathscr{C}_{e,R}^{[\mathfrak{a}]} := \{\phi \in \mathscr{C}_{e,R} \mid \phi(F^e_*\mathfrak{a}) \subset \mathfrak{a}\}.$$

That is, $\phi \in \mathscr{C}_{e,R}$ belongs to $\mathscr{C}_{e,R}^{[\mathfrak{a}]}$ iff there is a (necessarily unique) map $\bar{\phi} = \phi/\mathfrak{a} \in \mathscr{C}_{R/\mathfrak{a}}$ making the following diagram commutative

$$\begin{tikzcd}
F^e_* R \arrow[r, "\phi"] \arrow[d, two heads] & R \arrow[d, two heads] \\
F^e_* R/\mathfrak{a} \arrow[r, "\bar{\phi}"] & R/\mathfrak{a}
\end{tikzcd}$$

Observe that there's a canonical homomorphism

$$\mathscr{C}_R^{[\mathfrak{a}]} \longrightarrow \mathscr{C}_{R/\mathfrak{a}}$$

whose image we denote as

$$\bar{\mathscr{C}}_R^{[\mathfrak{a}]}.$$

In degree e, this image can be interpreted as the maps $F^e_* R/\mathfrak{a} \longrightarrow R/\mathfrak{a}$ admitting a lift to a map $F^e_* R \longrightarrow R$. If R admits a p-basis, then

$$\mathscr{C}_R^{[\mathfrak{a}]} \cong \bigoplus_{e \in \mathbb{N}} \mathfrak{a}^{[q]} : \mathfrak{a}$$

and

$$\bar{\mathscr{C}}_R^{[\mathfrak{a}]} \cong \bigoplus_{e \in \mathbb{N}} \frac{\mathfrak{a}^{[q]} : \mathfrak{a}}{\mathfrak{a}^{[q]}}$$

where (graded) multiplication on the right-hand side is given by

$$x \cdot y := x^{q'} y \in \mathfrak{a}^{[qq']} : \mathfrak{a}$$

if $x \in \mathfrak{a}^{[q]} : \mathfrak{a}$ and $y \in \mathfrak{a}^{[q']} : \mathfrak{a}$.

Example 6.8. Let R be a normal integral domain and P be a prime divisor over R. Show that

$$\mathscr{C}_R^P = \mathscr{C}_R^{[R(-P)]}.$$

Prove that the same hold if P is a reduced effective divisor, *i.e.* P is an effective divisor with coefficients in $\{0, 1\}$.

Example 6.9. Let R be an A-algebra. It then comes equipped with a diagonal morphism

$$R \otimes_A R \xrightarrow{r \otimes s \mapsto rs} R$$

More generally, we have an n-th diagonal morphism

$$R^{\otimes n} := R^{\otimes_A n} \longrightarrow R,$$

which has a kernel $\mathfrak{d}_n$. We define $\mathscr{D}_R^{(n)} \subset \mathscr{C}_R$ as the Cartier algebra of n-*diagonally compatible maps*. That is, $\mathscr{D}_R^{(n)} \subset \mathscr{C}_R$ is the image of

$$\mathscr{C}_{R^{\otimes n}}^{[\mathfrak{d}_n]} \longrightarrow \mathscr{C}_R.$$

Definition 6.10 (Cartier modules)**.** Let $\mathscr{C}$ be a Cartier algebra over a ring R. A *Cartier ($\mathscr{C}$-)module* is a left $\mathscr{C}$-module that is finitely generated as an R-module under the induced action by R. In other words, a Cartier module is an f.g. R-module M together with a homomorphism of Cartier algebras

$$\mathscr{C} \to \mathscr{C}_M.$$

A homomorphism of Cartier modules is a homomorphism of R-modules $\phi\colon M \to N$ such that the following diagram commutes

$$\begin{tikzcd} F^e_* M \arrow[r, "\kappa"] \arrow[d, "F^e_*\phi"'] & M \arrow[d, "\phi"] \\ F^e_* N \arrow[r, "\kappa"] & N \end{tikzcd}$$

for all $\kappa \in \mathscr{C}_e$ and all $e \in \mathbb{N}$.

Exercise 6.4. Let $\mathscr{C}$ be a Cartier algebra over R. Show that the category of Cartier $\mathscr{C}$-modules is an abelian category.

Example 6.11. Let $\mathscr{C} = R\{\kappa\}/(r\kappa = \kappa r^q)$ be a free Cartier algebra. Note that a Cartier module is the same things as a pair (M, ϕ) where M finitely generated R-module and $\phi \in \operatorname{Hom}_R(F^e_* M, M)$. In that case, we may refer to the pair (M, ϕ) as a Cartier module.

Remark 6.12. There is an useful notion of Cartier module in the literature that consists of an R-module M, an invertible R-module L, and an R-linear map $\phi\colon F^e_*(M \otimes L) \to M$ (for some $e \in \mathbb{N}$).

Example 6.13. The ring R itself is a Cartier $\mathscr{C}$-module for $\mathscr{C} = \mathscr{C}^{\Delta}_R, \mathscr{C}^{\mathfrak{a}^t}_R, \mathscr{D}^{(n)}_R, \ldots$

Example 6.14. $R/\mathfrak{a}$ is a Cartier $\mathscr{C}^{[\mathfrak{a}]}_R$-module.

Observe that $\mathscr{C}_+$ is a (two-sided) ideal of $\mathscr{C}$ and so the following makes sense:

Lemma 6.15 (Blickle–Gabber)**.** *Let $\mathscr{C}$ be a Cartier algebra over R and M be a Cartier module. Then, the descending sequence of (Cartier) submodules*

$$M \supset \mathscr{C}_+ M \supset \mathscr{C}^2_+ M \supset \mathscr{C}^3_+ M \supset \cdots$$

stabilizes.

Proof. Set

$$Z_n \coloneqq \operatorname{Supp}(\mathscr{C}^n_+ M / \mathscr{C}^{n+1}_+ M)$$

which is a closed subset of $\operatorname{Spec} R$ (as R is noetherian and M is f.g. as an R-module).[38] Our goal is to show that $Z_n = \emptyset$ for all $n \gg 0$. Observe that

$$Z_0 \supset Z_1 \supset Z_2 \supset \cdots$$

Since $\operatorname{Spec} R$ is a noetherian topological space, this sequence must stabilize. Moreover, we may then assume that

$$Z \coloneqq Z_0 = Z_1 = Z_2 = \cdots .$$

[38]Recall that the support of an R-module is the subset $\operatorname{Supp} M \coloneqq \{\mathfrak{p} \in \operatorname{Spec} R \mid M_{\mathfrak{p}} \neq 0\}$. When R is noetherian and M is f.g., we can say that $\operatorname{Supp} M$ is closed and in fact equal to $V(\operatorname{Ann}_R M)$. Further, $M = 0$ iff $\operatorname{Supp} M = \emptyset$.

Suppose for the sake of contradiction that $Z \neq \emptyset$. Let $\mathfrak{p} \in Z$ be the generic point of one of its irreducible components (*i.e.* $\mathfrak{p}$ is a minimal prime in Z or say a minimal prime of the radical ideal cutting Z out). By localizing at $\mathfrak{p}$, we may assume that R is local with maximal ideal $\mathfrak{p}$ and that $Z = \{\mathfrak{p}\}$ is a singleton.

Now, note that the case $n = 0$ implies that

$$\mathfrak{p}^k M \subset \mathscr{C}_+ M$$

for some $k \in \mathbb{N}$. In particular, for any $x \in \mathfrak{p}^k$ we have that

$$x^2 M \subset x\mathfrak{p}^k M \subset x\mathscr{C}_+ M \subset \sum_{0 \neq e \in \mathbb{N}} \mathscr{C}_e x^q M \subset \mathscr{C}_+(x^2 M)$$

By iterating this, we get that

$$x^2 M \subset \mathscr{C}_+^n M$$

for all $n \in \mathbb{N}$. Therefore, letting $\mu \coloneqq \mu(\mathfrak{p})$, we conclude that

$$\mathfrak{p}^{k(\mu-1)} M \subset \mathscr{C}_+^n M$$

for all $n \in \mathbb{N}$. Hence, the original sequence stabilizes if and only if so does the sequence

$$M/\mathfrak{p}^{k(\mu-1)}M \supset \mathscr{C}_+ M/\mathfrak{p}^{k(\mu-1)}M \supset \mathscr{C}_+^2 M/\mathfrak{p}^{k(\mu-1)}M \supset \mathscr{C}_+^3 M/\mathfrak{p}^{k(\mu-1)}M \supset \cdots$$

Observe that this sequence stabilizes as $M/\mathfrak{p}^{k(\mu-1)}M$ is a module of finite length; which yields the sought contradiction. □

Definition 6.16 (Nilpotent Cartier modules)**.** Let $\mathscr{C}$ be a Cartier algebra over R. A Cartier module M is *nilpotent* if $\mathscr{C}_+^n M = 0$ for all $n \gg 0$. We say that two Cartier modules M and N are *nil-isomorphic* of there is a homomorphism of Cartier modules $M \longrightarrow N$ whose kernel and cokernel are nilpotent as Cartier modules.

Exercise 6.5. Prove that nilpotent Cartier modules form a Serre subcategory, *i.e.* a full abelian subcategory that is closed under extensions.

Remark 6.17 (Cartier crystals). The above means that we may localize the category of Cartier $\mathscr{C}$-modules at its subcategory of nilpotent ones. This means that one formally inverts nil-isomoprhims to make them isomorphisms. This localization yields the so-called category of *Cartier crystals*. To be more precise, the objects of the category of Cartier $\mathscr{C}$-crystals are the exact same as the category of Cartier $\mathscr{C}$-modules. What changes are the morphisms. A morphism of Cartier crystals $M \longrightarrow N$ (here, M and N are Cartier $\mathscr{C}$-modules) is an equivalence class (aka left fraction) of diagrams of homomorphisms of Cartier modules

$$M \longleftarrow M' \rightarrow N$$

where $M' \longrightarrow M$ is a nil-isomorphism. Or, to be more precise,

$$\operatorname{Hom}_{\mathrm{Crys}}(M, N) \coloneqq \operatorname{colim}_{M' \longrightarrow M} \operatorname{Hom}_{\mathrm{Cart}}(M', N)$$

where the colimit runs over all nil-isomorphisms $M' \longrightarrow M$. It turns out that Cartier crystals then form an abelian category. However, I'm not sure just yet how much we want to use this pretty formalism. I'll provide more detail as we go and I feel we need them. But the point for now is that all we do here is about Cartier crystals rather than Cartier modules, *i.e.* we regard nil-potent Cartier structures as worthless.

Corollary 6.18. *Let $\mathscr{C}$ be a Cartier algebra over R and M be a Cartier module. There is a unique Cartier submodule*

$$\sigma(M) = \sigma(M, \mathscr{C}) \subset M$$

such that:

(a) The quotient $M/\sigma(M)$ is nilpotent (i.e. $\sigma(M) \subset M$ is an isomorphism of Cartier crystals).
(b) $\mathscr{C}_+ M = M$ (i.e. $\sigma(M)$ admits no nilpotent quotientts).

Proof. Take

$$\sigma(M) \coloneqq \mathscr{C}_+^n M, \quad n \gg 0$$

to be the stable element in Lemma 6.15. □

Definition 6.19 (F-purity of Cartier modules)**.** Let $\mathscr{C}$ be a Cartier R-algebra. One says that a Cartier $\mathscr{C}$-module M is *F-pure* if $\sigma(M) = M$. When referring to $M = R$ and $\mathscr{C} \subset \mathscr{C}_R$ some Cartier subalgebra such as $\mathscr{C} = \mathscr{C}_R^{\Delta}, \mathscr{C}_R^{\mathfrak{a}^t}, \mathscr{C}_R^{\phi} \dots$ we may simply say that $(R, \mathscr{C})$ or well that (R, Δ), $(R, \mathfrak{a}^t)$, (R, ϕ), etc are F-pure.

Exercise 6.6. Prove that a Cartier module M is F-pure iff $\mathscr{C}_+ M = M$ iff it admits no nilpotent quotients.

Exercise 6.7. Let $\mathscr{C}$ be a Cartier algebra over R and M be a Cartier module. Prove that if M is F-pure then $\operatorname{Ann} M$ is a radical ideal.

Next, we gotta verify that the above notion of F-purity matches the one we already had.

Proposition 6.20. *Let $\mathscr{C}$ be a Cartier algebra over R and suppose that R is a Cartier $\mathscr{C}$-module. Then, $(R, \mathscr{C})$ is F-pure iff there is $e > 0$ and $\kappa \in \mathscr{C}_e$ such that $\kappa(R) = R$.*

Proof. Note that, by definition, the F-purity of $(R, \mathscr{C})$ means that there are $0 \neq e_1, \dots, e_n \in \mathbb{N}$ and $\kappa_i \in \mathscr{C}_{e_i}$ such that

$$\sum_{i=1}^{m} \kappa_i(R) = R$$

The question is how to bring m down to 1. For this, let's set

$$e \coloneqq e_1 \cdots e_m.$$

Claim 6.21. $\sum_{i=1}^{m} \kappa^{e/e_i}(R) = R.$

Proof of the claim. This equality can be checked locally. Localizing

$$\sum_{i=1}^{m} \kappa_i(R) = R$$

at $\mathfrak{p}$ let us conclude that

$$\kappa_i(R_{\mathfrak{p}}) = R_{\mathfrak{p}}$$

for some $i = i_{\mathfrak{p}} \in \{1, \dots, m\}$ (as the sum of proper ideals in a local ring is proper). But then

$$\kappa_i^k(R_{\mathfrak{p}}) = R_{\mathfrak{p}}$$

for all $k \geq 1$ and so for $k = e/e_i$. In particular, the required equality holds at all $\mathfrak{p} \in \operatorname{Spec} R$. □

In particular, there are $r_i \in R$ such that

$$\sum_{i=1}^{m} \kappa^{e/e_i}(r_i) = 1.$$

Hence, we may take $\kappa := \sum_{i=1}^{m} \kappa^{e/e_i} \cdot r_i \in \mathscr{C}_e$ as $\kappa^{e/e_i} \in \mathscr{C}_e$ for all i. $\square$

Exercise 6.8. Let R be a Cohen–Macaulay ring with Cartier operator $\kappa_R \colon F_*\omega_R \longrightarrow \omega_R$. Show that R is F-injective iff (ω_R, κ_R) is F-pure.

Exercise 6.9. Let $\mathscr{C}$ be a Cartier algebra over R and M be a Cartier module. Show that

$$W^{-1}\sigma(M, \mathscr{C}) = \sigma(W^{-1}M, \mathscr{C}) = \sigma(W^{-1}M, W^{-1}\mathscr{C}).$$

Exercise 6.10. Prove that of $\mathscr{C}$ is a Cartier algebra over R making R a Cartier module then the non-F-pure locus of $(R, \mathscr{C})$ is cut out by the ideal $\sigma(R)$.[39] Moreover, show that ideals $\sigma(R, \mathscr{C})$ and $\mathscr{C}_+R$ cut out the same closed subset of $\operatorname{Spec} R$, *i.e.* they have the same radical ideal.

Let $\mathscr{C}$ be a Cartier R-algebra and $\mathfrak{a}$ be an ideal. Observe that

$$\mathscr{C}\mathfrak{a} = \bigoplus_{e \in \mathbb{N}} \mathscr{C}_e \mathfrak{a}$$

is a a two-sided ideal. In particular, the quotient

$$\mathscr{C}/\mathscr{C}\mathfrak{a}$$

is a Cartier algebra. To be precise, it is a Cartier algebra over $R/\mathfrak{a}$. Moreover, note that a Cartier $\mathscr{C}$-module M such that $\mathfrak{a}M = 0$ (*i.e.* $\mathfrak{a} \subset \operatorname{Ann}_R M$) can be regarded as a Cartier $\mathscr{C}/\mathscr{C}\mathfrak{a}$-module and vice versa.

Let $i \colon \operatorname{Spec} R/\mathfrak{a} \longrightarrow \operatorname{Spec} R$ be the spectrum of $R \longrightarrow R/\mathfrak{a}$. The image of i identifies $\operatorname{Spec} R/\mathfrak{a}$ with $V(\mathfrak{a})$. Recall that this gives two functors i_* and $i^!$ as well as an adjointness relation $i_* \dashv i^!$. Observe that the unit

$$\epsilon_N \colon M \longrightarrow i^! i_* N$$

is a natural isomorphism on the category of $R/\mathfrak{a}$-modules. On the other hand, the co-unit

$$\eta_M \colon i_* i^! M \longrightarrow M$$

is given by

$$\operatorname{Hom}_R(R/\mathfrak{a}, M) \xrightarrow{\cong\colon \phi \mapsto \phi(1)} \{m \in M \mid m\mathfrak{a} = 0\} \subset M.$$

The next exercise shows that if M is a Cartier module thenη_M is naturally a homomorphism of Cartier modules,

Exercise 6.11. Let $\mathscr{C}$ be a Cartier algebra over R and M be a Cartier module. Show that

$$i^! M := \operatorname{Hom}_R(R/\mathfrak{a}, M) \cong \{m \in M \mid m\mathfrak{a} = 0\} \subset M$$

is a Cartier submodule for all ideals $\mathfrak{a}$. Conclude that

$$H^0_{\mathfrak{a}}(M) \subset M$$

is a Cartier submodule too. Conclude that these are further Cartier $\mathscr{C}/\mathscr{C}\mathfrak{a}$-modules.

[39]This is why $\sigma(R)$ is often referred to as the non-F-pure ideal, which I personally regard as a poor name.

Exercise 6.12. Let $\mathscr{C}$ be a Cartier algebra over R and M be a Cartier module. Suppose that M is F-pure and that $\operatorname{Supp} M \subset V(\mathfrak{a})$. Show that

$$\operatorname{Hom}_R(R/\mathfrak{a}, M) \subset M$$

is an equality.

We can use all of the above to obtain the following simple but crucial observation.

Proposition 6.22. *Let $\mathscr{C}$ be a Cartier algebra over R and $\mathfrak{a} \subset R$ be an ideal. The functor $i^! \coloneqq \operatorname{Hom}_R(R/\mathfrak{a}, -)$ induces an equivalence of categories from the category of F-pure Cartier $\mathscr{C}$-modules with support inside $V(\mathfrak{a})$ to the category of F-pure Cartier $\mathscr{C}/\mathscr{C}\mathfrak{a}$-modules.*

Proof. We just argued that the adjointness relation $i_* \dashv i^!$ defines an equivalence of categories when we restrict ourselves to F-pure Cartier modules. ☐

Definition 6.23 (Test Modules First Definition)**.** Let $\mathscr{C}$ be a Cartier algebra over R and M be a Cartier module. We let

$$\tau(M) = \tau(M, \mathscr{C}) \subset M$$

denote the smallest Cartier submodule of M that agrees generically with $\sigma(M)$ (*i.e.* $N_\eta = \sigma(M)_\eta$ for all minimal primes of $\operatorname{Ann} M$). We refer to $\tau(M)$ as the *test module* of $(M, \mathscr{C})$.

The question is, when do test modules exist? Test modules are known to exists in any of the following three cases:

(a) R is essentially of finite type over an F-finite field,
(b) $\mathscr{C}$ is generated by one single operator, or
(c) $M \subset R$.

Open Problem 6.13. Find the largest setup for which test modules exists.

Later on we'll focus on (a generalization) of the third case and time permitting then the second one. For now, I want you to know that the existence of these objects is intimately related to finiteness results on Cartier crystals.

Theorem 6.24. *Let $\mathscr{C}$ be a Cartier algebra over R and M be a Cartier module. Then, the lattice of F-pure submodules of M (under intersection and sum) satisfies DCC (as well as ACC) iff $\tau(N)$ exists for all Cartier submodules $N \subset M$.*

Proof. Let's show the DCC condition by induction on $\dim R$ (which is $< \infty$ as R is F-finite). The case $\dim R = 0$ is clear.

Let

$$M \supset M_0 \supset M_1 \supset M_2 \supset \cdots$$

be a descending chain of F-pure Cartier submodules of M. Since $\operatorname{Spec} R$ is a noetherian topological space, the corresponding sequence of supports stabilizes. In particular, after truncating, we may write (and assume)

$$Z \coloneqq \operatorname{Supp} M_i, \quad \forall i \in \mathbb{N}.$$

But, since the M_i are F-pure, this further means that

$$\mathfrak{a} \coloneqq \operatorname{Ann} M_i, \quad \forall i \in \mathbb{N}.$$

Here, $\mathfrak{a}$ is a readical ideal. By Proposition 6.22, we may replace R by $R/\mathfrak{a}$ and so assume that R is reduced and that the M_i are supported everywhere.

Likewise, since it suffices to show that the sequence stabilize after restricting at each irreducbile component, we may further assume that R is an integral domain.

Moreover, we may assume (after truncating again) that all the M_i have the same rank (i.e. they're all generically equal).

With all those reductions in place, we proceed as follows. Let use that $\tau(M_0)$ exists. By definition of test modules, $\tau(M_0) \subset M_i$ for all i (in fact, $\tau(M_0) = \tau(M_i)$). In particular, it suffices tp show that the following sequence of Cartier $\mathscr{C}$-modules stabilizes:

$$M_0/\tau(M_0) \subset M_1/\tau(M_0) \subset M_2/\tau(M_0) \subset M_3/\tau(M_0) \subset \cdots$$

Notice that these modules are generically zero! Therefore, they're supported on a proper closed subset of $\operatorname{Spec} R$. Here, we simply let induction to do its thing.

The converse statement is left as an exercise. $\square$

Theorem 6.25 (Existence of test modules)**.** *Let $\mathscr{C}$ be a Cartier algebra over R and M be a Cartier $\mathscr{C}$-module. Then, $\tau(M)$ exists in any the following cases:*

(a) R is essentially of finite type over an F-finite field,
(b) $\mathscr{C}$ is generated by one single operator, or
(c) $M \subset R$.

As it was mentioned above, we'll come back to the prove of this later on. Assuming the existence of test modules, then we can conclude the following.

Corollary 6.26. *Let $(M, \mathscr{C})$ be an F-pure Cartier module. Assume test modules exist. Then, the set of radical ideals*

$$\{\operatorname{Ann} M/N \mid N \subset M \text{ is a Cartier } \mathscr{C}\text{-submodule}\}$$

is:

(a) finite,
(b) closed under finite unions and intersections,
(c) closed under prime decomposition.

Proof. We only show the finiteness result leaving the rest to the reader. The proof is an induction on $\dim R$. The case $\dim R = 0$ is trivial. We do the standard reductions. First, we may assume that $\operatorname{Supp} M = \operatorname{Spec} R$. Since there are finitely many irreducible components (R is noetherian), we may further assume that R is irreducible with generic point η.

Suppose that $\operatorname{Ann} M/N \neq \sqrt{0}$. Then, $M_\eta = N_\eta$ and so $\tau(M) \subset N$. Moreover,

$$\operatorname{Ann} M/N \supset \operatorname{Ann} M/\tau(M) =: \mathfrak{a} \neq \sqrt{0}.$$

We can then apply the inductive hypothesis on $R/\mathfrak{a}$ to conclude. $\square$

Exercise 6.14. Prove the statements (b) and (c) in Corollary 6.26.

Corollary 6.27. *Let $(R, \mathscr{C})$ be an F-pure pair. Then, the set of $\mathscr{C}$-ideals*

$$I(R, \mathscr{C}) \coloneqq \{\mathfrak{a} \in I(R) \mid \mathscr{C}\mathfrak{a} \subset \mathfrak{a}\}$$

is a a finite sub-lattice of the lattice of radical ideals that is closed under prime decomposition.

There's a very satisfying way to think of the about finiteness results:

Corollary 6.28 (Finiteness in Cartier crystals)**.** *Let $\mathscr{C}$ be a Cartier algebra over R. Every Cartier crystal has finite length in the category of Cartier crystals. In particular, every Cartier crystals has only finitely many Cartier subcrystals.*

Definition 6.29 (F-regular cartier module)**.** We say that a Cartier module $(M,\mathscr{C})$ is F-*regular* if it is F-pure and moreover any Cartier submodule $N \subset M$ that agrees with M generically is equal to M.[40]

Exercise 6.15. Prove that a Cartier modules $(M,\mathscr{C})$ is F-regular iff $\tau(M,\mathscr{C}) = M$.

Exercise 6.16. Show that $(\tau(M,\mathscr{C}),\mathscr{C})$ is always F-regular.

Exercise 6.17. Show that $(R,\mathscr{C})$ is F-regular if and only if R is reduced and for all nonzerodivisor $r \in R$ there is $e \gg 0$ and $\kappa \in \mathscr{C}_e$ such that $\kappa(r) = 1$. *Hint: Adapt the proof of Proposition 6.20.*

Exercise 6.18. Concluded that if R is a domain, then $(R,\mathscr{C})$ is F-regular iff it is F-pure and simple as Cartier module. Further, show that if $(R,\mathscr{C})$ is simple as a Cartier module then its F-purity amounts to $\mathscr{C}_+R \neq 0$ (*i.e.* $\sigma(R,\phi) \neq 0$). Prove that if R is a domain but not a field (*i.e.* simple as an R-module), then $(R,\mathscr{C})$ is F-regular iff it is simple as a Cartier module.

Definition 6.30. Let R be a ring with Cartier operator $\kappa_R \colon F_*\omega_R \longrightarrow \omega_R$. We say that R is F-rational if it is Cohen–Macaualay and (ω_R,κ_R) is F-regular.

In particular, we see that the most important elements in $I(R,\mathscr{C})$ are those that are prime ideals.

Definition 6.31 (Centers of F-purity)**.** Let $(R,\mathscr{C})$ be a Cartier module. A center of F-purity is an element $\mathfrak{p}$ in $I(R,\phi) \cap \operatorname{Spec} R$ such that $\mathfrak{p} \neq V(\sigma(R,\phi))$, *i.e.* $\mathfrak{p}$ belongs to the F-pure locus of $(R,\mathscr{C})$. We denote the set of centers of F-purity of $(R,\mathscr{C})$ by $\operatorname{Schpec}(R,\mathscr{C})$.

We'll see that spectrum of centers of F-purity is finite. Unless $(R,\mathscr{C})$ is F-pure, this doesn't follow from Corollary 6.26. In fact, Corollary 6.26 only tell us that the set of centers of F-purity contained in $\sigma(R,\mathscr{C})$ are finite. To get the full finiteness result we need a more refined type of test ideal—the so-called test ideals along a center of F-purity.

Definition 6.32 (Test ideals along a center of F-purity)**.** Let $(R,\mathscr{C})$ be a Cartier module and $\mathfrak{p} \in \operatorname{Schpec}(R,\phi)$. The *test ideal of* $(R,\mathscr{C})$ *along* $\mathfrak{p}$, denoted by $\tau_{\mathfrak{p}}(R,\mathscr{C})$, is the smallest ideal in $I(R,\mathscr{C})$ not contained in $\mathfrak{p}$.

Theorem 6.33. *Test ideals along centers of F-purity exist.*

Proof. Postponed to a later section. □

6.1. **Some standard applications to local cohomology.** In this subsection, let's fix $\mathscr{C}$ to be $R\{\kappa\}/(r\kappa = \kappa r^p)$ and a ring R. In particular, Cartier modules are pairs (M,κ) where M is a finitely generated R-module and $\kappa \in \operatorname{Hom}_R(F_*M,M)$ The following is the dual notion of Cartier module.

Definition 6.34 (F-module)**.** An *F-module* M is an R-module M equipped with a map $F = \phi \in \operatorname{Hom}_R(M,F_*M)$. We often may ask for M to be artinian.

Example 6.35. The ring $R/\mathbb{F}_p$ is itself and F-module and so are its local cohomology modules $H^i_{\mathfrak{a}}(R)$.

[40]Recall that agreeing generically means that $N_\eta = M_\eta$ at every minimal prime η in the support of M.

As we've seen before, Grothendieck duality for finite morphisms let us see that if $(R, \mathfrak{m}, \mathscr{k})$ is a local ring with injective hull of the residuel field E, then there is a natural isomorphism

$$F_* \circ \mathscr{D} \cong \mathscr{D} \circ F_*$$

where $\mathscr{D} \coloneqq \operatorname{Hom}_R(-, E)$. Then, Matlis duality takes the following shape.

Theorem 6.36 (Matlis duality for Cartier and F-modules)**.** *Let $(R, \mathfrak{m}, \mathscr{k})$ be a complete local ring. Then the Matlis functor $\mathscr{D}$ induces an anti-equivalence between the category of Cartier modules and (artinian) F-modules.*

Remark 6.37. More traditionally, F-modules are denoted as $F\colon M \longrightarrow M$ where F is such that $F(rm) = r^p F(m)$.

As an application, we have:

Corollary 6.38. *Let $F\colon M \longrightarrow M$ be an F-module. Then:*

(a) The descending chain

$$\ker F \subset \ker F^2 \subset \cdots$$

stabilizes.

(b) Every chain

$$N_1 \subset N_2 \subset \cdots \subset M$$

of F-submodules eventually have F-nilpotent quotients.

(c) Up to F-nilpotency, M has only finitely many F-submodules.

Exercise 6.19. Let $(R, \mathfrak{m}, \mathscr{k})$ be a normal Cohen–Macaulay F-injective local ring. Show that there are only finitely many *F-submodules* of $H^{\dim R}_{\mathfrak{m}}(R)$, *i.e.* R-submodules $M \subset H^{\dim R}_{\mathfrak{m}}(R)$ such that $F(M) \subset M$.

Exercise 6.20. Show that if $(R, \mathfrak{m}, \mathscr{k})$ is quasi-Gorenstein (*i.e.* $H^d_{\mathfrak{m}}(R) = E_R(\mathscr{k})$) and F-split then $H^d_{\mathfrak{m}}(R)$ has only finitely many F-submodules.

We introduce next Lyubeznik's notion of F-finite module on regular rings:

Definition 6.39. Let R be a regular ring. Let N be a finitely generated R-module together with a map $\gamma\colon N \longrightarrow F^*N$. Consider $\mathscr{N}$ to be the limit of the directed system

$$N \xrightarrow{\gamma} F^*N \xrightarrow{F^*\gamma} F^{2,*}N \xrightarrow{F^{2,*}\gamma} F^{3,*}N \longrightarrow \cdots$$

with induced isomorphism

$$\gamma\colon \mathscr{N} \xrightarrow{\cong} F^*\mathscr{N}.$$

The pair $(\mathscr{N}, \gamma)$ is referred to as an *F-finite module.*[41] You know what, let's refer to them as *Lyubeznik modules.*

Remark 6.40. Technically speaking, the definition of Lyubeznik module doesn't need R to be regular for it to makes sense. However, it's the setup in which it is nice. This is needed to show that the category of Lyubenik modules is a Serre subcategory of R-modules.

[41]This is a terrible name. Maybe a better name is Lyubeznik module.

Example 6.41. Let R be a local ring. Then, the local cohomology modules $H^i_{\mathfrak{a}}(R)$ are Lyubeznik modules. Indeed, recall that

$$H^i_{\mathfrak{a}}(M) = \varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^i_R(R/\mathfrak{a}^n, M)$$

However, if $\mathfrak{a}$ is generated by n elements then

$$\mathfrak{a}^{nq} \subset \mathfrak{a}^{[q]} \subset \mathfrak{a}^q$$

which means that

$$H^i_{\mathfrak{a}}(M) = \varinjlim_{e\in\mathbb{N}} \operatorname{Ext}^i_R(R/\mathfrak{a}^{[q]}, M).$$

Moreover,

$$F^{e,*}(R/\mathfrak{a}) = R/\mathfrak{a}^{[q]}$$

In particular, we get a canonical (quotient) map

$$R/\mathfrak{a}^{[q]} \longrightarrow F^*(R/\mathfrak{a}^{[q]}) = R/\mathfrak{a}^{[pq]}$$

Now, since R is regular and so F^e is flat, we then have

$$F^* \operatorname{Ext}^i_R(R/\mathfrak{a}^{[q]}, R) = \operatorname{Ext}^i_{F^*R}(F^*(R/\mathfrak{a}^{[q]}), F^*R) = \operatorname{Ext}^i_R(R/\mathfrak{a}^{[pq]}, R)$$

and applying the functor $\operatorname{Ext}^i_R(-, R)$ to

$$R/\mathfrak{a} \longrightarrow R/\mathfrak{a}^{[p]} = F^*R/\mathfrak{a}$$

then gives a map

$$\gamma\colon N := \operatorname{Ext}^i_R(R/\mathfrak{a}, R) \longrightarrow F^*N$$

realizing $H^i_{\mathfrak{a}}(R)$ as an F-finite R-module (*i.e.* Lyubeznik R-module) with root $N = \operatorname{Ext}^i_R(R/\mathfrak{a}, R)$.

In general, there's a way to cook up a Lyubeznik module out of a Cartier one. For this we need the following observation.

Exercise 6.21. Let $\theta\colon R \longrightarrow S$ be a finite flat extension with corresponding map on spectra $f\colon \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$. Prove that the canonical map

$$\operatorname{Hom}_R(S, R) \otimes_R M \xrightarrow{\sigma\otimes m\mapsto(s\mapsto\sigma(s)m)} \operatorname{Hom}_R(S, M)$$

induces a natural isomorphism of functors

$$\omega_f \otimes f^* \xrightarrow{\cong} f^!,$$

where it's worth recalling that $\omega_f := f^!R$. Conclude that if R is a regular ring then we have a natural isomorphism

$$\omega_f \otimes F^* \xrightarrow{\cong} F^!$$

Let's further recall how the Cartier operator $\kappa\colon F_*\omega_R \longrightarrow \omega_R$ was defined. First, we recall that we have an adjointness relation

$$F_* \dashv F^!$$

Then, under the assumption that there's an isomorphism

$$\omega_R \xrightarrow{\cong} F^!\omega_R$$

such an isomorphism has to be adjoint to something like $F_*\omega_R \longrightarrow \omega_R$, which we lovingly call Cartier operator.

Now, if R is regular so that $\omega_R = R(K_R)$ for some Cartier divisor K_R, we can then say that

$$\omega_f \otimes F^*\omega_R \cong \omega_R$$

and so that we have a natural isomorphism

$$\omega_R \otimes F^*(\omega_R^{-1} \otimes M) \xrightarrow{\cong} F^! M.$$

where $\omega_R^{-1} = R(-K_R)$.

In this way, if we have a Cartier module $F_*M \longrightarrow M$ this is adjoint (*i.e.* fancy way to say the exact same thing) to a map

$$M \longrightarrow F^! M = \omega_R \otimes F^*(\omega_R^{-1} \otimes M).$$

which we can tensor by ω_R^{-1} to get a map

$$\gamma \colon \omega_R^{-1} \otimes M \longrightarrow F^*(\omega_R^{-1} \otimes M)$$

and by taking the limit $e \to \infty$ we get a Lyubeznik module.

The above process yields a functor $\mathscr{L}$ from the category of Cartier modules to the one of Lyubeznik modules

Exercise 6.22. Show that $\mathscr{L}(\kappa_R \colon F_*\omega_R \longrightarrow \omega_R)$ is $F \colon R_{\mathrm{perf}} \xrightarrow{\cong} R_{\mathrm{perf}}$.

Exercise 6.23. Show that $\mathscr{L}$ is essentially surjective.

Exercise 6.24. Show that a nilpotent Cartier module maps to the zero Lyubeznik module.

In particular, $\mathscr{L}$ induces an essentially surjective functor from the category of Cartier crystals to the one of Lyubeznik modules. In fact:

Theorem 6.42. *The functor $\mathscr{L}$ yields an equivalence of categories from the category of Cartier crystals to the one of Lyubeznik modules.*

Proof. Let $\gamma \colon \mathscr{N} \longrightarrow F^*\mathscr{N}$ be a Lyubeznik module. We may consider all the maps $\gamma \colon N \longrightarrow F^*N$ giving rise to this Lyubeznik module. Then, they correspond uniquely to a bunch of Cartier modules $F_*(N \otimes \omega_R) \longrightarrow N \otimes \omega_R$. The point is that these are nil-isomorphic as Cartier modules. I'LL ADD THE DETAILS AT SOME POINT. $\square$

Corollary 6.43. *Let R be a regular (and F-finite of course) ring. Then:*

(a) Lyubeznik modules have finite length in the category of Lyubeznik modules.
(b) A lyubeznik module as finitely many Lyubeznik submodules.

Corollary 6.44. *Let R be a regular (and F-finite of course) ring. Then $H^i_{\mathfrak{a}}(R)$ has finitely many associated primes.*

7. Existence of Test Elements and Test ideals

In this section, we finally argue why test ideals exists in the case $M = R$ and $\mathscr{C}$ is principal. We also discuss the existence of test ideals along centers of F-purity.

7.1. **General generalities.** We start off with the following key observation that the existence of test modules is equivalent to the generic F-regularity of F-pure Cartier modules.

Theorem 7.1. *Let $(M,\mathscr{C})$ be a Cartier module over R. Then, $\tau(M,\mathscr{C})$ exists if and only if there is a principal open subset $D(c) = \operatorname{Spec} R_c$ such that:*

(a) the open $D(c) \cap \operatorname{Supp} \sigma(M,\mathscr{C})$ is dense in $\operatorname{Supp} \sigma(M,\mathscr{C})$ (i.e. c avoids every minimial prime of $\operatorname{Supp} \sigma(M,\mathscr{C})$), and

(b) the localized pair $(\sigma(M,\mathscr{C}),\mathscr{C})_c$ is F-regular.

Moreover, in that case, $\tau(M,\mathscr{C})$ is the Cartier submodule of $(M,\mathscr{C})$ generated by $c^t\sigma(M,\mathscr{C})$ for any $0 \neq t \in \mathbb{N}$.

Definition 7.2. We refer to any such $c \in R$ in Theorem 7.1 as a *test element of* $(R,\mathscr{C})$.

Proof of Theorem 7.1. Since M and $\sigma(M)$ share the same test module, we may replace M by $\sigma(M)$ and assume that M is F-pure. Then, as usual, we may replace R by $R/\operatorname{Ann}_R M$ and assume that M is supported everywhere and that R is reduced.

Assume now that there is a dense open subset $D(c) \subset \operatorname{Spec} R$ such that $\tau(M_c,\mathscr{C}_c) = M_c$ (*i.e.* we have generic F-regularity).

Claim 7.3. *If $N \subset M$ is a Cartier submodule such that $N_\eta = M_\eta$ at every minimal prime then $cM \subset N$.*

Proof of the claim. The inclusion $N_c \subset M_c$ gotta be an equality by the assumed generic F-regularity on $D(c)$. Since Cartier modules are finitely generated, this means that $c^t(M/N) = 0$ for some $t \gg 0$. However, $\operatorname{Ann}_R M/N$ is a radical ideal! Hence $c(M/N) = 0$; as required. □

Then, the Cartier submodule of M generated by cM is a test module.

Conversely, suppose that $\tau(M,\mathscr{C})$ exists. Take c to be any element of $\operatorname{Ann}_R M/\tau(M,\mathscr{C})$ avoiding every minimal prime. Indeed, then we have:

$$M_c = \tau(M,\mathscr{C})_c = \tau(M_c,\mathscr{C}_c)$$

and we're done. □

Remark 7.4. In articular, this means that to show that $\tau(M,\mathscr{C})$ exists, we may assume that R is a regular domain and that M is free of finite rank! Our next reduction is that $\mathscr{C}$ can be taken to be finitely generated. To this end, we need the following two lemmas.

Lemma 7.5. *Let $(M,\mathscr{C})$ be a Cartier module over R. There is a finitely generated Cartier subalgebra $\mathscr{C}' \subset \mathscr{C}$ such that $\sigma(\mathscr{C}) = \sigma(\mathscr{C}')$. Moreover, if $\tau(M,\mathscr{C}')$ exists then so does $\tau(M,\mathscr{C}')$ and one has*

$$\tau(M,\mathscr{C}') \subset \tau(M,\mathscr{C}'') = \tau(M,\mathscr{C})$$

for some finitely generated Cartier algebra $\mathscr{C}' \subset \mathscr{C}'' \subset \mathscr{C}$.

Exercise 7.1. Let $(M,\mathscr{C})$ be a Cartier module with Cartier submodule N. Let $\mathscr{C}' \subset \mathscr{C}$ be a Cartier subalgebra. Prove that $\sigma(N,\mathscr{C}') \subset \sigma(M,\mathscr{C})$ and that when it is an equality then if exists $\tau(N,\mathscr{C}')$ then so does $\tau(M,\mathscr{C})$ and moreover $\tau(N,\mathscr{C}') \subset \tau(M,\mathscr{C})$.

Lemma 7.6. *Let $(M,\mathscr{C})$ be a Cartier module. There is a finitely generated Cartier subalgebra $\mathscr{C}' \subset \mathscr{C}$ such that $\sigma(M,\mathscr{C}') = \sigma(M,\mathscr{C})$. Moreover, if $\tau(M,\mathscr{C}')$ exists then so does $\tau(M,\mathscr{C})$ and in fact*

$$\tau(M,\mathscr{C}') \subset \tau(M,\mathscr{C}'') = \tau(M,\mathscr{C})$$

where $\mathscr{C}' \subset \mathscr{C}'' \subset \mathscr{C}$ is a finitely generated Cartier subalgebra.

Proof. May assume that $(M, \mathscr{C})$ is F-pure. In particular

$$\mathscr{C}_+ M = \sum_{e \geq 1} \sum_{\kappa \in \mathscr{C}_e} \kappa(M) = M$$

and since M is f.g. we can write

$$\mathscr{C}_+ M = \sum_{i=1}^{k} \kappa_i(M)$$

for some $\kappa_i \in \mathscr{C}_{e_i}$ and $0 \neq e_1, \ldots, e_k \in \mathbb{N}$. Let $\mathscr{C}'$ be the Cartier subalgebra of $\mathscr{C}$ generated by $\kappa_1, \ldots, \kappa_k$.

To conclude, one let c be a test element of $\tau(M, \mathscr{C})$ so that

$$\tau(M, \mathscr{C}) = \sum_{e \in \mathbb{N}} \mathscr{C}_e c M$$

and use that $\tau(M, \mathscr{C})$ is f.g. to write

$$\tau(M, \mathscr{C}) = \sum_{i=1}^{l} \kappa'_i(cM)$$

for some $\kappa'_i \in \mathscr{C}_{e'_i}$ and $e'_1, \ldots, e'_l \in \mathbb{N}$. Let $\mathscr{C}''$ be the Cartier subalgebra generated by $\mathscr{C}'$ and $\kappa'_1, \ldots, \kappa'_l$. □

Exercise 7.2. In the setup of Lemma 7.6 and its proof, show that $\tau(M, \mathscr{C}') = \tau(M, \mathscr{C}'')$ as $\mathscr{C}''$ was constructed to be finitely generated over $\mathscr{C}'$.

Corollary 7.7. *In proving the existence of* $\tau(M, \mathscr{C})$*, we may assume that:*

(a) R is a regular domain,
(b) $\mathscr{C}$ is finitely generated,
(c) M is a free R-module (of finite rank),
(d) $(M, \mathscr{C})$ is F-pure,

and then show that $(R, \mathscr{C})$ is generically F-regular: i.e. there is $0 \neq c \in R$ such that $(M, \mathscr{C})_c$ is F-regular.

Proposition 7.8. *Let $(M, \mathscr{C})$ be an F-pure Cartier module over a regular domain R such that $\mathscr{C}$ is finitely generated and M is a free R-module of rank $\mu < \infty$. Suppose either that:*

(a) $\mu \leq 1$, or
(b) $\mathscr{C}$ is principally generated.

Then, $(M, \mathscr{C})$ is generically F-regular.

Proof. The first step is to explain why the case (b) can be reduced to case (a). So let's take a Cartier module structure

$$\phi\colon F^e_* R^{\oplus \mu} \longrightarrow R^{\oplus \mu}.$$

Observe that

$$\phi \in \operatorname{Hom}(F^e_* R^{\oplus \mu}, R^{\oplus \mu}) \cong \operatorname{Hom}(F^e_* R, R)^{\oplus \mu \times \mu}$$

which means that ϕ can be thought of as a matrix maps

$$\phi = (\phi_{i,j})_{\mu \times \mu}, \quad \phi_{i,j} \in \operatorname{Hom}(F^e_* R, R).$$

Now, chose a map $\Phi \in \mathrm{Hom}_R(F^e_* R, R)$ such that

$$K \xrightarrow{1 \mapsto 1\otimes\Phi = \Phi_K} K \otimes_R \mathrm{Hom}_R(F^e_* R, R) = \mathrm{Hom}_K(F^e_* K, K)$$

is an isomorphism. That is, choose Φ to be a generic Frobenius trace.[42] This means that the R-linear map

$$R \xrightarrow{1\mapsto\Phi} \mathrm{Hom}_R(F^e_* R, R)$$

is injective and its cokernel is torsion (as everytrhing is f.g.). In particular, we may replace R by R_c for some $0 \neq c \in R$ and assume that it's an isomorphism (*i.e.* take c any non-zero annihilator of the cokernel). That is, we may assume that Φ is a Frobenius trace (or say a Cartier operator) over R.

In this way, there's a (uniquely determined) square R-matrix $A = (a_{i,j})_{\mu\times\mu}$ such that

$$\phi_{i,j} = \Phi^e \cdot a_{i,j}, \quad \forall i, j.$$

In other words, $\phi\colon F^e_* R^{\oplus\mu} \longrightarrow R^{\oplus\mu}$ is going to be equal to the composition

$$F^e_* R^{\oplus\mu} \xrightarrow{F^e_* A} R^{\oplus\mu} \xrightarrow{(\Phi^e)^{\oplus\mu}} R^{\oplus\mu}$$

In particular, $\ker A$ and so $\operatorname{coker} A$ are Cartier modules too. More precisely, $\ker A$ is a Cartier submodule of $M = R^{\oplus\mu}$ whereas $\operatorname{coker} A$ is a quotient of it. In fact, since ϕ annihilates $\ker A$ straight away merciless, we see that M and $\operatorname{coker} A$ are nil-isomorphic. In particular, in proving that M is generically F-regular we may replace it by $\operatorname{coker} A$. This means that we may assume that A is injective and so that $\det A \neq 0$ (*i.e.* that A is generically injective). With this reduction in place, we may shrinking replacing R by $R_{\det A}$ and assume that A is actually an isomorphism. In doing so, we may assume that A is the identity, or which is the same assume that

$$\phi = (\Phi^e)^{\oplus\mu}.$$

In particular, we can actually assume that $\mu = \phi^e$ and so that $\mu = 1$. That's how we can reduce to the case in which $M = R$.

Assume (a), which is to say that we may assume that $M = R$ and $\mathscr{C} \subset \mathscr{C}_R$. As before, we may shrink R enough to say that

$$\mathscr{C}_R = \mathscr{C}_R^{\Phi}$$

for some Frobenius trace $\Phi\colon F_* R \longrightarrow R$. But then, we may take any Frobenius splitting $\phi \in \mathscr{C}_e$ and write

$$\phi = \Phi^e \cdot r$$

for a uniquely determined $0 \neq r \in R$. Replacing R by R_r, we then may assume that $\phi = \Phi^e$ and moreover that

$$\mathscr{C} = \mathscr{C}_R^{\Phi^e}.$$

The rest of the proof is left as an exercise (*e.g.* solve the following two exercises!). ☐

Exercise 7.3. Prove that

$$\tau(M, \phi) = \tau(M, \phi^n)$$

for all $n > 0$ (and all Cartier R-module $\phi\colon F^e_* M \longrightarrow M$). *Hint:*

Exercise 7.4. Prove that a regular ring R is F-regular.

Remark 7.9. I don't see how we used the hypothesis $\mathscr{C}$ is finitely generated. Do you?

[42] The fact we can do this rely heavily on R being an F-finite integral domain.

Corollary 7.10. *Let $(M, \mathscr{C})$ be a Cartier module over R such that either $M \subset R$ or $\mathscr{C}$ is principal. Then, $\tau(M, \mathscr{C})$ exists.*

7.2. **Test ideals along radical Cartier ideals.** Let's start by fixing a Cartier module $(R, \mathscr{C})$. Recall that we let $I(R, \mathscr{C})$ the set of $\mathscr{C}$-ideals or say the ideals of $(R, \mathscr{C})$. These are the $\mathscr{C}$-submodules of R. For notation ease, we may think of $\mathscr{C}$ as a Cartier subalgebra $\mathscr{C} \subset \mathscr{C}_R$. Then, $\mathfrak{a} \in I(R, \mathscr{C})$ iff $\phi(F^e_* \mathfrak{a}) \subset \mathfrak{a}$ for all $\phi \in \mathscr{C}_e$ and all $e \in \mathbb{N}$. Recall that $I(R, \mathscr{C})$ is a sublattice of $I(R)$.

Exercise 7.5. Show that if $\mathfrak{a} \in I(R, \mathscr{C})$ and $\mathfrak{b} \in I(R)$ then $\mathfrak{a} : \mathfrak{b} \in I(R, \mathscr{C})$. Conclude that $\operatorname{Ass}_R(R/\mathfrak{a}) \subset I(R, \mathscr{C})$ ans $\sqrt{\mathfrak{a}} \in I(R, \phi)$.

Remark 7.11. Recall that if $(R, \mathscr{C})$ is F-pure then $I(R, \phi)$ is a finite set of radical ideals.

Definition 7.12 (Non-degeneracy and pure F-regularity)**.** With notation as above, fix $\mathfrak{a} \in I(R, \mathscr{C})$. Then:

(a) We say that $(R, \mathscr{C})$ is *non-degenerate along* $\mathfrak{a}$ if the minimal primes of $\mathfrak{a}$ are centers of F-purity of $(R, \mathscr{C})$.
(b) We say that $(R, \mathscr{C})$ is *purely F-regular along* $\mathfrak{a}$ if it's non-degenerate along $\mathfrak{a}$ and every proper $\mathscr{C}$-ideal is contained in at least one minimal prime of $\mathfrak{a}$. In that case, we refer to the minimal primes of $\mathfrak{a}$ as the *maximal centers of F-purity of* $(R, \mathscr{C})$.
(c) We say that R is *purely F-regular along* $\mathfrak{a}$ if $(R, \mathscr{C}_R^{[\mathfrak{a}]})$ is purely F-regular along $\mathfrak{a}$.

Remark 7.13. Notice that $(R, \mathscr{C})$ is F-regular iff it is purely F-regular along (0). Also, by prime avoidance, non-degeneracy along $\mathfrak{a}$ means that $\mathscr{C}_+ R$ is not contained in the union of the associated associated primes of $R/\mathfrak{a}$.

Exercise 7.6. Prove that if $(R, \mathscr{C})$ is F-regular along $\mathfrak{a}$ than R and $R/\mathfrak{a}$ are F-regular. Show that the converse holds if R is regular.

Exercise 7.7. Prove that $(R, \mathscr{C})$ is non-degenerate along $\mathfrak{a}$ if and only if there's $e > 0$ and $\phi \in \mathscr{C}_e$ such that $\phi(F^e_* R)$ avoids every minimal prime of $\mathfrak{a}$. *Hint: Adapt the proof of Proposition 6.20, cf. Exercise 6.17.*

Exercise 7.8. If $(R, \mathscr{C})$ is purely F-regular along $\mathfrak{a}$ then $(R, \mathscr{C})$ is F-pure and so $\mathfrak{a}$ is radical. Moreover, the minimal primes of $\mathfrak{a}$ are pairwise coprime.

We aim to prove the following:

Theorem 7.14. *Let $(R, \mathscr{C})$ be a pair as before and $\mathfrak{a} \in I(R, \mathscr{C})$ be radical such that $(R, \mathscr{C})$ is non-degenerate along $\mathfrak{a}$. Then, there exists*

$$c \notin \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R R/\mathfrak{a}} \mathfrak{p} =: Q$$

such that for all $r \notin Q$ there is $e > 0$ and $\phi \in \mathscr{C}_e$ such that

$$\phi(F^e_* r) = c.$$

Furthermore, the ideal

$$\tau_{\mathfrak{a}}(R, \mathscr{C}) \coloneqq \mathscr{C}_+ c = \sum_{0 \neq e \in \mathbb{N}} \mathscr{C}_+ c$$

has the following three properties:

(a) $\tau_{\mathfrak{a}}(R,\mathscr{C}) \not\subset Q$,
(b) $\tau_{\mathfrak{a}}(R,\mathscr{C}) \in I(R,\mathscr{C})$, *and*
(c) if $\mathfrak{b} \in I(R,\mathscr{C})$ *is such that* $\mathfrak{b} \not\subset Q$ *then* $\tau_{\mathfrak{a}}(R,\mathscr{C}) \subset \mathfrak{b}$.

Definition 7.15 (Test elements). We refer to $\tau_{\mathfrak{a}}(R,\mathscr{C})$ as the *test ideal of* $(R,\mathscr{C})$ *along* $\mathfrak{a}$. This is the smallest $\mathscr{C}$-ideal not contained in any minimal prime of $\mathfrak{a}$. We refer to any such element c in Theorem 7.14 as a *principal test element of* $(R,\mathscr{C})$ *along* $\mathfrak{a}$. Likewise, the elements of $\tau_{\mathfrak{a}}(R,\mathscr{C}) \setminus Q$ are called *test elements of* $(R,\mathscr{C})$ *along* $\mathfrak{a}$. When $\mathfrak{a}=(0)$ all the above applies but we drop "along $\mathfrak{a}$."

Exercise 7.9. Show that if c is a principal test element of $(R,\mathscr{C})$ along $\mathfrak{a}$ then so is $\phi(F^e_* c)$ for all $\phi \in \mathscr{C}_e$ and all $e \in \mathbb{N}$. Conclude that test elements are sums of principal test elements and that the test ideal $\tau_{\mathfrak{a}}(R,\mathscr{C})$ is generated by principal test elements (of $(R,\mathscr{C})$ along $\mathfrak{a}$).

Exercise 7.10. Let $(R,\mathscr{C})$ be a pair as before and $\mathfrak{a} \in I(R,\phi)$ be radical such that $(R,\mathscr{C})$ is non-degenerate along $\mathfrak{a}$. Show the following:

(a) Note that
$$I(R,\mathscr{C}) = \{\mathfrak{b} \in I(R,\mathscr{C}) \mid \mathfrak{b} \subset Q\} \sqcup \{\mathfrak{b} \in I(R,\mathscr{C}) \mid \tau_{\mathfrak{a}}(R,\mathscr{C}) \subset \mathfrak{b}\}$$
and conclude that $(R,\mathscr{C})$ is purely F-regular along $\mathfrak{a}$ iff $\tau_{\mathfrak{a}}(R,\mathscr{C}) = R$.

(b) Let b be a test element of $(R,\mathscr{C})$ along $\mathfrak{a}$. Show that $\tau_{\mathfrak{a}}(R,\mathscr{C}) = \mathscr{C}b = Rb + \mathscr{C}_+ b$.

(c) Prove that $(R,\mathscr{C})$ is purely F-regular along $\mathfrak{a}$ if and only if $1 \in R$ is a principal test element.

Exercise 7.11. Let $(R,\mathscr{D} \subset \mathscr{C})$ be a pair and $\mathfrak{a} \in I(R,\mathscr{C})$ be radical such that $(R,\mathscr{D})$ is non-degenerate along $\mathfrak{a}$. Show that $\tau_{\mathfrak{a}}(R,\mathscr{D}) \subset \tau_{\mathfrak{a}}(R,\mathscr{C})$.

Exercise 7.12. Let $(R,\mathscr{C})$ be a pair and $\mathfrak{a} \in I(R,\mathscr{C})$ be radical such that $(R,\mathscr{C})$ is non-degenerate along $\mathfrak{a}$. Let $\mathfrak{p}_1,\dots,\mathfrak{p}_k$ be the minimal primes of $\mathfrak{a}$. Show that if $\tau_{\mathfrak{p}_i}(R,\mathscr{C})$ exists for all $i=1,\dots,k$ then so does $\tau_{\mathfrak{a}}(R,\mathscr{C})$ and moreover
$$\tau_{\mathfrak{a}}(R,\mathscr{C}) = \sum_{i=1}^{k} \tau_{\mathfrak{p}_i}(R,\mathscr{C}).$$

Exercise 7.13. Let $(R,\mathscr{C})$ be a pair and $\mathfrak{a} \in I(R,\mathscr{C})$ be radical such that $(R,\mathscr{C})$ is non-degenerate along $\mathfrak{a}$. Let $W \subset R$ be a multiplicatively closed subset such that $W \cap Q = \emptyset$. Prove that
$$W^{-1}\tau_{\mathfrak{a}}(R,\mathscr{C}) = \tau_{W^{-1}\mathfrak{a}}(W^{-1}R, W^{-1}\mathfrak{a}).$$
What happens if $W \cap Q \neq \emptyset$?

Exercise 7.14. Prove that if ϕ and ψ are such that $\Delta_\phi = \Delta_\psi$ then they share the same centers of F-purity and $\tau_{\mathfrak{p}}(R,\phi) = \tau_{\mathfrak{p}}(S,\psi)$ for any common center of F-purity.

7.3. **Proof of Theorem 7.14.** Throughout this subsection, we let $(R,\mathscr{C})$ be a pair as before and $\mathfrak{a} \in I(R,\mathscr{C})$ be radical such that $(R,\mathscr{C})$ is non-degenerate along $\mathfrak{a}$. Also, we let Q be the union of all the associated primes of $R/\mathfrak{a}$ as before. We start off with the following:

Lemma 7.16. *There exists* $\gamma \in R \setminus Q$ *such that:*

(a) If $(1) \neq \mathfrak{b} \in I(R_\gamma,\mathscr{C}_\gamma)$ *then* $\mathfrak{b} \subset \mathfrak{a}R_\gamma$.
(b) $(R_\gamma,\mathscr{C}_\gamma)$ *is* F*-pure.*

Proof. Let $\pi\colon R \longrightarrow R/\mathfrak{a}$ be the quotient homomorphism. Note that a test element of $(R/\mathfrak{a}, \mathscr{C})$ does the job for (a) as $R/\mathfrak{a}$ is reduced. That is, since $R/\mathfrak{a}$ is reduced and $\mathscr{C}|_{R/\mathfrak{a}}$ is non-degenerate, then we may pick a nonzerodivisor $\pi(\gamma)$ in $\boldsymbol{\tau}(R/\mathfrak{a}, \mathscr{C})$. Then, $\gamma \in R$ satisfies the condition (a).

To ensure that (b) also holds, pick $\gamma' \in R \setminus Q$ in the image of $\phi(F^e_* R)$ for some e and $\phi \in \mathscr{C}_e$ using Exercise 7.7. Then, $\gamma\gamma'$ does the job. □

Lemma 7.17. *There exists $\gamma \in R \setminus Q$ such that for all $r \in R \setminus Q$ there exists $n \in \mathbb{N}$, $0 \neq e \in \mathbb{N}$, and $\phi \in \mathscr{C}_e$ such that*

$$\gamma^n = \phi(F^e_* r).$$

Proof. As observed many times before at this point, it suffices to show that there is $\gamma \in R \setminus Q$ such that

$$(\mathscr{C}_+ r)_\gamma = R_\gamma$$

for all $r \in R \setminus Q$. I claim that the γ granted by Lemma 7.16 does the job. Indeed, it'd suffice to prove that $(\mathscr{C}_+ r)_\gamma$ is a $\mathscr{C}_\gamma$-ideal but not contained in $\mathfrak{a}R_\gamma$. One readily sees that $(\mathscr{C}_+ r)_\gamma$ is a $\mathscr{C}_\gamma$-ideal. To see that it isn't contained in $\mathfrak{a}R_\gamma$, we may proceed as follows.

Since $(R/\mathfrak{a}, \mathscr{C})_\gamma$ is F-regular and $r \in R/\mathfrak{a}$ avoids all minimal primes, we know that there is $\phi \in \mathscr{C}_e$ for some $e \gg 0$ such that

$$(\mathscr{C}_+ r)_\gamma \ni \phi(F^e_* r) = 1 + x$$

for some $x \in \mathfrak{a}R_\gamma$. Hence, $(\mathscr{C}_+ r)_\gamma$ cannot be contained in $\mathfrak{a}R_\gamma$. □

Lemma 7.18. *Let $\phi \in \mathscr{C}_e$, $c \in R$, and $b \in \phi(F^e_* cR)$. Then, $b^2 \in \phi^n(F^{ne}_* cR)$ for all $0 \neq n \in \mathbb{N}$.*

Proof. The proof is done by induction on n where the case case is given by hypothesis. Observe that:

$$b^2 \in b\phi(F^e_* cR) = \phi(F^e_* b^q cR) \subset \phi(F^e_* b^2 cR) \subset \phi(F^e_* \phi^n(F^{ne}_* cR)c) \subset \phi^{n+1}(F^{(n+1)e}_* cR)$$

where the penultimate inclusion is the inductive hypothesis. □

Proof of Theorem 7.14. Let γ be given by Lemma 7.17. In particular, there are m and $\phi \in \mathscr{C}_e$ with m and e minimal such that

$$\gamma^m = \phi(F^e_* 1).$$

Then, by Lemma 7.18, it follows that

$$\gamma^{2m} \in \phi^n(F^{en}_* R)$$

for all $n > 0$. We claim that we can take

$$c \coloneqq \gamma^{3m}.$$

Indeed, for $r \in R \setminus Q$, there will be $0 \neq m \in \mathbb{N}$ and $\phi' \in \mathscr{C}_{e'}$ such that

$$\gamma^{m'} = \phi'(F^{e'}_* r).$$

If $m' < 3m$ we're then done. Else, let's take n sufficiently large such that

$$m' < mq^n.$$

Then:

$$c \coloneqq \gamma^m \gamma^{2m} \in \gamma^m \phi^n(F^{ne}_* R) = \phi^n(F^{ne}_* \gamma^{mq^n} R) \subset \phi^n(F^{ne}_* \gamma^{m'} R) \subset (\phi^n \cdot \phi')(F^{ne+e'}_* rR),$$

which finishes the proof of the first part of the theorem (*i.e.* the existence of c as a principal test element). The rest is left to the diligent student as an exercise. $\square$

7.4. **The restriction theorem and finiteness of centers of F-purity.** Here, we study the important *restriction theorem* and use it together with the existence of test ideals along centers of F-purity to show finiteness of centers of F-purity.

Theorem 7.19 (Restriction theorem)**.** *Let $(R,\mathscr{C})$ be a pair and $\mathfrak{a} \in I(R,\mathscr{C})$ be radical such that $(R,\mathscr{C})$ is non-degenerate along $\mathfrak{a}$. Let $\mathfrak{b} \subset \mathfrak{a}$ be a $\mathscr{C}$-subideal such that $\mathscr{C}|_{R/\mathfrak{b}}$ is non-degenerate. Then,*

$$\tau_{\mathfrak{a}}(R,\mathscr{C})R/\mathfrak{b} = \tau_{\mathfrak{a}/\mathfrak{b}}(R/\mathfrak{b},\mathscr{C}|_{R/\mathfrak{b}}).$$

Exercise 7.15. Prove the restriction theorem.

Corollary 7.20. *Let $(R,\mathscr{C})$ be a Cartier. The spectrum of centers of F-purity* $\operatorname{Schpec}(R,\mathscr{C})$ *is a finite set.*

Proof. Suppose for the sake of contradiction that $\operatorname{Schpec}(R,\mathscr{C})$ is an infinite set. Since R has finite dimension (as it has always been assumed F-finite), there is a height $0 < h \leq \dim R$ such that the are infinitely many elements of $\operatorname{Schpec}(R,\mathscr{C})$ of height h. Let $S \subset \operatorname{Schpec}(R,\mathscr{C})$ be the center of F-purity of height h. Let $Z \subset \operatorname{Spec} R$ the Zariski closure of S, *i.e.* Z is the closed subset cutout by the radical ideal

$$\mathfrak{a} \coloneqq \bigcap_{\mathfrak{p}\in S} \mathfrak{p}.$$

Observe that $\mathfrak{a} \in I(R,\mathscr{C})$ and $(R,\mathscr{C})$ is non-degenerate along $\mathfrak{a}$. Therefore, we can cook up $\tau_{\mathfrak{a}}(R,\mathscr{C})$.

Moreover, notice that we may assume that

$$\operatorname{ht} \mathfrak{a} < h.$$

Indeed, there can only be finitely minimal primes of $\mathfrak{a}$ of height h. Just throw them away.

In particular, we can say that $\tau_{\mathfrak{a}}(R,\mathscr{C})$ is contained in every $\mathfrak{p} \in S$ and so in $\mathfrak{a}$. This, however, violates the restriction theorem. Cool. $\square$

If there are finitely many centers of F-purity attached to a given pair $(R,\mathscr{C})$, how does one find all of them? I'm glad you asked. Let's spend a bit of time on that.

8. The Cartier Core Map

Let's start off with the notion of transposition of an ideal along a q^{-1}-linear map. We still assume all rings to be F-finite and noetherian.

Definition 8.1. Let $\mathfrak{a} \in I(R)$ be an ideal of a ring R and $\phi \in \mathscr{C}_{e,R}$. Then:

(a) We set $\mathfrak{a}_\phi \coloneqq \phi(F^e_*\mathfrak{a})$.
(b) We define the the *ϕ-tranpose ideal* $\mathfrak{a}^\phi$ as the largest ideal $\mathfrak{b}$ such that $\mathfrak{b}_\phi \subset \mathfrak{a}$. That is,

$$\mathfrak{a}^\phi \coloneqq \{r \in R \mid (r)_\phi = \phi(F^e_* rR) = (\phi \cdot r)(F^e_*) \subset \mathfrak{a}\}.$$

(c) We say that ϕ is *non-degenerate along* $\mathfrak{a}$ if the image of ϕ is not contained in any minimal prime of $\mathfrak{a}$.

Exercise 8.1. Prove the following properties:

(a) Transpositions commute with localizations.
(b) Transposition commute with arbitrary intersections of ideals whereas $(-)_\phi$ conmutes with arbitrary sums.
(c)
$$\mathfrak{b} \subset \mathfrak{a}^\phi \Longleftrightarrow \mathfrak{b}_\phi \subset \mathfrak{a}$$
(d)
$$V(\mathfrak{b}_\phi) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{b} \subset \mathfrak{p}^\phi\}$$
(e)
$$\mathfrak{p}^\phi \neq R \Longleftrightarrow \phi(F^e_*) \not\subset \mathfrak{p}$$
(f) The locus of degeneracy of ϕ is
$$V(\sigma(R,\phi)) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p}^\phi = R\}.$$
and so coincides with the non-F-pure locus of (R,ϕ).
(g)
$$\mathfrak{a} \subset R \cap F^e_* \mathfrak{a}^\phi = \mathfrak{a} : \mathbf{1}_\phi$$
and the inclusion is an equality iff ϕ is non-degenerate along $\mathfrak{a}$.
(h) If ϕ is non-degenerate along $\mathfrak{a}$ then $\mathfrak{a} \subset \mathfrak{a}^\phi$.
(i)
$$\mathfrak{a}^{\phi\cdot\psi} = (\mathfrak{a}^\phi)^\psi$$
(j)
$$I(R,\phi) = \{\mathfrak{a} \in I(R) \mid \mathfrak{a}_\phi \subset \mathfrak{a}\} = \{\mathfrak{a} \in I(R) \mid \mathfrak{a} \subset \mathfrak{a}^\phi\}$$

Remark 8.2. The ideals $\mathfrak{a} \in I(R)$ such that $\mathfrak{a} = \mathfrak{a}^\phi$ are referred to as *fixed ϕ-ideals*. We may then conclude that non-degenerate ϕ-ideals are fixed. But there can be more; they happen to be rather mysterious. I myself must confess don't understand very well. I don't know what's their deal but they seem to matter in certain contexts.

Remark 8.3. Note that:
$$I(R,\mathscr{C}) = \bigcap_{e\in\mathbb{N}} \bigcap_{\phi\in\mathscr{C}_e} I(R,\phi)$$

Definition 8.4. Let's define two spectra. First, the *Cartier spectrum*
$$\operatorname{CSpec}(R,\mathscr{C}) \coloneqq I(R,\mathscr{C}) \cap \operatorname{Spec} R$$
And the one we've already defined, the *Schwede spectrum of centers of F-purity*
$$\operatorname{Schpec}(R,\mathscr{C}) \coloneqq \{\mathfrak{p} \in \operatorname{CSpec}(R,\mathscr{C}) \mid \bar{\phi}_\mathfrak{p} = \phi_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \neq 0\}$$
We're gonna endow them with the subspace Zariski topology from $\operatorname{Spec} R$. We'll write:
$$V_\mathscr{C}(\mathfrak{a}) \coloneqq \operatorname{CSpec}(R,\mathscr{C}) \cap V(\mathfrak{a})$$
and likewise
$$V^\circ_\mathscr{C}(\mathfrak{a}) \coloneqq \operatorname{Schpec}(R,\mathscr{C}) \cap V(\mathfrak{a}) = \operatorname{Schpec}(R,\mathscr{C}) \cap V_\mathscr{C}(\mathfrak{a})$$

Remark 8.5. The reason for the circle in the notation above is that

$$\mathrm{CSpec}(R,\mathscr{C}) = \mathrm{Schpec}(R,\mathscr{C}) \sqcup V(\sigma(R,\mathscr{C}))$$

and so $\mathrm{Schpec}(R,\mathscr{C})$ is the open given by intersection of $\mathrm{CSpec}(R,\mathscr{C})$ with the F-pure locus of $(R,\mathscr{C})$. Moreover,

$$\begin{aligned}(R,\mathscr{C}) \text{ is } F\text{-regular} &\iff \dim \mathrm{CSpec}(R,\mathscr{C}) = 0 \\ &\iff (R,\mathscr{C}) \text{ is } F\text{-pure and } \mathrm{Schpec}(R,\mathscr{C}) = \mathrm{Ass}\, R.\end{aligned}$$

Exercise 8.2. Let $(R,\mathscr{C})$ be a Cartier pair, $\mathfrak{a} \in I(R,\mathscr{C})$, and $W \subset R$ be a multiplicatively closed subset.

(a) Prove that extension of ideals along the quotient $\pi \colon R \longrightarrow R/\mathfrak{a}$ induces an isomorphism of lattices
$$\{\mathfrak{b} \in I(R,\mathscr{C}) \mid \mathfrak{a} \subset \mathfrak{b}\} \xrightarrow{\cong} I(R/\mathfrak{a},\mathscr{C}/\mathfrak{a})$$
which restricts to homeomorphisms
$$V_{\mathscr{C}}(\mathfrak{a}) \xrightarrow{\cong} \mathrm{CSpec}(R/\mathfrak{a},\mathscr{C}/\mathfrak{a})$$
and
$$V^{\circ}_{\mathscr{C}}(\mathfrak{a}) \xrightarrow{\cong} \mathrm{Schpec}(R/\mathfrak{a},\mathscr{C}/\mathfrak{a})$$

(b) Show that extension of ideals along the localization $R \longrightarrow W^{-1}R$ gives a homomorphism of lattices
$$I(R,\phi) \longrightarrow I(W^{-1}R, W^{-1}\mathscr{C})$$
which restricts to an isomorphism
$$\{\mathfrak{p} \in \mathrm{CSpec}(R,\mathscr{C}) \mid \mathfrak{p} \cap W = \emptyset\} \xrightarrow{\cong} \mathrm{CSpec}(W^{-1}R, W^{-1}\mathscr{C})$$
and
$$\{\mathfrak{p} \in \mathrm{Schpec}(R,\mathscr{C}) \mid \mathfrak{p} \cap W = \emptyset\} \xrightarrow{\cong} \mathrm{Schpec}(W^{-1}R, W^{-1}\mathscr{C}).$$

Exercise 8.3. Let (R,ϕ) be a Cartier pair where R is a normal integral domain and $\phi \neq 0$. Let P be a prime divisor on $\mathrm{Spec}\, R$. Show that:

(a) $\Delta_\phi \geq P \iff R(-P) \in \mathrm{CSpec}(R,\phi)$.
(b) The elements of $\mathrm{CSpec}(R,\phi)$ oh height 1 are exactly those corresponding to prime divisors supporting Δ_ϕ with coefficient ≥ 1.
(c) The elements of $\mathrm{Schpec}(R,\phi)$ oh height 1 are exactly those corresponding to prime divisors supporting Δ_ϕ with coefficient $= 1$.
(d) If (R,ϕ) is F-pure then the coefficients of Δ_ϕ belong to $[0,1] \cap \mathbb{Z}_{(p)}$.

Definition 8.6 (Cartier Core)**.** Let $(R,\mathscr{C})$ be a pair and $\mathfrak{a} \in I(R)$. The *Cartier core* of $\mathfrak{a}$ is the $\mathscr{C}$-ideal
$$\kappa_{\mathfrak{a}}(R,\mathscr{C}) \coloneqq \bigcap_{0 \neq e \in \mathbb{N}} \bigcap_{\phi \in \mathscr{C}_e} \mathfrak{a}^{\phi}$$
Likewise, we define the image ideal of $(R,\mathscr{C})$ along $\mathfrak{a}$ as
$$\lambda_{\mathfrak{a}}(R,\mathscr{C}) \coloneqq \sum_{0 \neq e \in \mathbb{N}} \sum_{\phi \in \mathscr{C}_e} \mathfrak{a}_{\phi}$$

Exercise 8.4. Let $(R,\mathscr{C})$ be a pair and $\mathfrak{a}, \mathfrak{b} \in I(R)$. Prove the following properties:

(a) $W^{-1}\kappa_{\mathfrak{a}}(R,\mathscr{C}) = \kappa_{W^{-1}\mathfrak{a}}(W^{-1}R, W^{-1}\mathscr{C})$

(b) $\kappa_{-}(R,\mathscr{C})$ commutes with arbitrary intersections whereas $\lambda_{-}(R,\mathscr{C})$ does it with arbitrary sums.
(c) $\mathfrak{a} \subset \kappa_{\mathfrak{b}}(R,\mathscr{C}) \iff \lambda_{\mathfrak{a}}(R,\mathscr{C}) \subset \mathfrak{b}$
(d) $V(\lambda_{\mathfrak{a}}(R,\mathscr{C})) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{a} \subset \kappa_{\mathfrak{p}}(R,\mathscr{C})\}$
(e) $\kappa_{\mathfrak{a}}(R,\mathscr{C}) \neq \mathbf{1} \iff \mathscr{C}_{+}R \not\subset \mathfrak{a}$
(f) If $\mathfrak{b} \in I(R,\mathscr{C})$ is contained in $\mathfrak{a}$ then $\mathfrak{b} \subset \kappa_{\mathfrak{a}}(R,\mathscr{C})$.
(g) If $(R,\mathscr{C})$ is non-degenerate along $\mathfrak{a}$ then $\kappa_{\mathfrak{a}}(R,\mathscr{C}) \subset \mathfrak{a}$ and so $\kappa_{\mathfrak{a}}(R,\mathscr{C})$ is the largest $\mathscr{C}$-ideal contained in $\mathfrak{a}$. In particular, if $\mathfrak{a}$ is further a $\mathscr{C}$-ideal then $\kappa_{\mathfrak{a}}(R,\mathscr{C}) = \mathfrak{a}$.

Exercise 8.5. Let $(R,\mathscr{C})$ be a pair and $\mathfrak{a} \in I(R,\mathscr{C})$. Note that $\sigma(\mathfrak{a},\mathscr{C})$ is the largest fixed $\mathscr{C}$-ideal inside $\mathfrak{a}$. Prove that

$$V(\sigma(\mathfrak{a},\mathscr{C})) = V(\lambda_{\mathfrak{a}}(R,\mathscr{C})) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{a} \subset \kappa_{\mathfrak{p}}(R,\mathscr{C})\}.$$

Conclude that $(R,\mathscr{C})$ is non-degenerate along $\mathfrak{a}$ if and only if $\mathfrak{a}$ is radical and $\mathfrak{a} = \kappa_{\mathfrak{a}}(R,\mathscr{C})$.

Exercise 8.6. Conclude that the F-pure locus of $(R,\mathscr{C})$ is

$$\mathscr{U}(R,\mathscr{C}) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \kappa_{\mathfrak{p}}(R,\mathscr{C}) \neq 1\}$$

Exercise 8.7. Let $(R,\mathscr{C})$ be a pair and $\mathfrak{p} \in \mathscr{U}(R,\mathscr{C})$ in the F-pure locus of $(R,\mathscr{C})$. Prove that $\kappa_{\mathfrak{p}}(R,\mathscr{C}) \in \operatorname{Schpec}(R,\mathscr{C})$. Conclude that:

$$V(\lambda_{\mathfrak{a}}(R,\mathscr{C})) = \{\mathfrak{p} \in \mathscr{U}(R,\mathscr{C}) \mid \kappa_{\mathfrak{p}}(R,\mathscr{C}) \in V(\mathfrak{a})\} \sqcup V(\sigma(R,\mathscr{C}))$$

for all $\mathfrak{a} \in I(R)$.

Definition 8.7. Let $(R,\mathscr{C})$ be a pair and $\mathfrak{a} \in I(R)$. We define the Cartier radical of $\mathfrak{a}$ as

$$\rho_{\mathfrak{a}}(R,\mathscr{C}) := \mathfrak{a} + \lambda_{\mathfrak{a}}(R,\mathscr{C}) \in I(R,\mathscr{C}),$$

which is the smalles $\mathscr{C}$-ideal that contains $\mathfrak{a}$. Likewise, we define

$$\beta_{\mathfrak{a}}(R,\mathscr{C}) := \mathfrak{a} \cap \kappa_{\mathfrak{a}}(R,\mathscr{C}),$$

the largest $\mathscr{C}$-ideal contained in $\mathfrak{a}$.

Exercise 8.8. Let $(R,\mathscr{C})$ be a pair and $\mathfrak{a} \in I(R)$. Prove the following properties:
(a) $\mathfrak{a} \in I(R,\mathscr{C}) \iff \beta_{\mathfrak{a}}(R,\mathscr{C}) = \mathfrak{a} \iff \rho_{\mathfrak{a}}(R,\mathscr{C}) = \mathfrak{a}$
(b) If $(R,\mathscr{C})$ is non-degenerate along $\mathfrak{a}$ then $\beta_{\mathfrak{a}}(R,\mathscr{C}) = \kappa_{\mathfrak{a}}(R,\mathscr{C})$, $\rho_{\mathfrak{a}}(R,\mathscr{C}) = \lambda_{\mathfrak{a}}(R,\mathscr{C})$, and these are radical ideals.
(c) β commutes with intersections whereas ρ with sums.

Proposition 8.8. *Let $(R,\mathscr{C})$ be a pair and $\mathfrak{a},\mathfrak{b} \in I(R)$. If $\mathfrak{p} \in \operatorname{Spec} R$ then*

$$\beta_{\mathfrak{p}}(R,\mathscr{C}) = \begin{cases} \kappa_{\mathfrak{p}}(R,\phi) & \text{if } \mathfrak{p} \in \mathscr{U}(R,\mathscr{C}), \\ \mathfrak{p} & \text{otherwise} \end{cases} \in \operatorname{CSpec}(R,\mathscr{C}).$$

Therefore, this defines a mapping

$$\beta \colon \operatorname{Spec} R \xrightarrow{\mathfrak{p} \mapsto \beta_{\mathfrak{p}}(R,\mathscr{C})} \operatorname{CSpec}(R,\mathscr{C})$$

that retracts the inclusion $\operatorname{CSpec}(R,\mathscr{C}) \subset \operatorname{Spec} R$*. Moreover*

$$\mathfrak{a} \subset \beta_{\mathfrak{b}}(R,\mathscr{C}) \iff \rho_{\mathfrak{a}}(R,\mathscr{C}) \subset \mathfrak{b}$$

and in particular:

(a) β *is continuous as*

$$V(\rho_{\mathfrak{a}}(R,\mathscr{C})) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \beta_{\mathfrak{p}}(R,\mathscr{C}) \in V(\mathfrak{a})\} = \beta^{-1}V(\mathfrak{a})$$

(b) $\sqrt{\rho_{\mathfrak{a}}(R,\mathscr{C})} = \bigcap_{\mathfrak{p}\in V_{\mathscr{C}}(\mathfrak{a})} \mathfrak{p}$

(c) $V_{\mathscr{C}}(\mathfrak{a}) = V_{\mathscr{C}}(\mathfrak{b}) \iff \sqrt{\rho_{\mathfrak{a}}(R,\mathscr{C})} = \sqrt{\rho_{\mathfrak{b}}(R,\mathscr{C})}$

(d) The mapping $\mathfrak{a} \mapsto V_{\mathscr{C}}(\mathfrak{a})$ *is a bijection between radical ideals in* $I(R,\mathscr{C})$ *and closed subsets of* $\operatorname{CSpec}(R,\mathscr{C})$*.*

(e) The topology on $\operatorname{Spec} R$ *whose closed subsets are* $\{V(\mathfrak{a}) \subset \operatorname{Spec} R \mid \mathfrak{a} \in I(R,\mathscr{C})\}$ *is the coarsest topology on* $\operatorname{Spec} R$ *such that* β *is continous.*

Proof. Exercise to the reader. □

Definition 8.9 (Cartier–Zariski topology)**.** The Cartier–Zariksi topology on $\operatorname{Spec} R$ is the one given by (e) in Proposition 8.8. We denote the corresponding topological space by $\operatorname{CZpec}(R,\mathscr{C})$ and we consider the continuous map

$$\beta = \beta(R,\mathscr{C})\colon\ \operatorname{CZpec}(R,\mathscr{C}) \xrightarrow{\mathfrak{p}\mapsto\beta_{\mathfrak{p}}} \operatorname{CSpec}(R,\mathscr{C}).$$

Exercise 8.9. Show that for all subset $S \subset \operatorname{Spec} R$ its Cartier–Zariski closure is given by

$$\overline{S}^{\mathrm{CZ}} = \overline{\beta(S)}^{\mathrm{Z}} = V\left(\bigcap_{\mathfrak{p}\in S} \beta_{\mathfrak{p}}\right)$$

Conclude the following equivalences for a fixed $\mathfrak{a} \in I(R)$:

$$\overline{V(\mathfrak{a})}^{\mathrm{CZ}} = V(\mathfrak{a}) \iff \beta(V(\mathfrak{a})) \subset V(\mathfrak{a}) \iff \sqrt{\mathfrak{a}} \in I(R,\mathscr{C}).$$

Exercise 8.10. Let $(R,\mathscr{C})$ be a Cartier pair. Show that

$$V^{\circ}_{\mathscr{C}}(\mathfrak{a}) = V^{\circ}_{\mathscr{C}}(\sigma(\mathfrak{a},\mathscr{C}))$$

for all $\mathfrak{a} \in I(R,\mathscr{C})$. Prove further that two fixed $\mathscr{C}$-ideals define the same closed subset in $\operatorname{Schpec}(R,\mathscr{C})$ if and only if they have the same radical. Conclude that $\mathscr{U}(R,\mathscr{C})$ is an open subset of $\operatorname{CZpec}(R,\mathscr{C})$ whose closed subsets are of the form $V(\mathfrak{a}) \cap \mathscr{U}(R,\mathscr{C})$ where $\mathfrak{a}$ is a fixed $\mathscr{C}$-ideal.

Exercise 8.11. Show that the fiber of

$$\beta = \beta(R,\mathscr{C})\colon\ \operatorname{CZpec}(R,\mathscr{C}) \xrightarrow{\mathfrak{p}\mapsto\beta_{\mathfrak{p}}} \operatorname{CSpec}(R,\mathscr{C}).$$

is given by

$$\beta^{-1}(\mathfrak{p}) = \begin{cases} V(\mathfrak{p}) \setminus V(\tau_{\mathfrak{p}}(R,\mathscr{C})) & \text{if } \mathfrak{p} \in \mathscr{U}(R,\mathscr{C}), \\ \{\mathfrak{p}\} & \text{otherwise.} \end{cases}$$

and so that they're locally closed. That is, if $\mathfrak{p} \in \mathscr{U}(R,\mathscr{C})$ is in the F-pure locus then $\beta^{-1}(\mathfrak{p})$ is the the purely F-regular locus along $\mathfrak{p}$ that's inside $V(\mathfrak{p})$.

On a topological space X, define the equivalence relation

$$x \sim y \iff \overline{\{x\}} = \overline{\{y\}}$$

and let

$$\alpha_X\colon X \longrightarrow X/\sim$$

be the corresponding quotient in the category of topological spaces; say Top.

Exercise 8.12. Given a map $f\colon Y \longrightarrow X$ in Top (*i.e.* a continuous map), show that there is an induced commutative diagram in Top:

$$\begin{tikzcd} X \arrow[d, "\alpha_X"'] & Y \arrow[l, "f"'] \arrow[d, "\alpha_Y"] \\ X/{\sim} & Y/{\sim} \arrow[l, "f/{\sim}"'] \end{tikzcd}$$

Conclude that there's a natural transformation

$$\alpha\colon \mathrm{id}_{\mathsf{Top}} \longrightarrow (-)/{\sim}$$

between the functors

$$\mathrm{id}_{\mathsf{Top}}, (-)/{\sim}\colon \mathsf{Top} \longrightarrow \mathsf{Top}.$$

Exercise 8.13. Show that if Y is an irreducible subset of a topological space X then so is $\overline{Y}$. Conclude that the closed subsets $\overline{\{x\}}$ are irreducible.

Terminology 8.10. One refers to elements $y \in \overline{\{x\}}$ as *specializations* of x and write $x \rightsquigarrow y$ for all $y \in \overline{\{x\}}$. One also says that x is a *generization* of y.

Exercise 8.14. Show that if X is a T_0 space then $x \rightsquigarrow y$ defines a partial order in X.

Definition 8.11. A topological space X is *quasi-sober* (resp. *sober*) if the mapping $x \mapsto \overline{\{x\}}$ is a surjection (resp. bijection) from X onto the irreducible closed subsets of X. A topological space is *quasi-Zariski* (resp. *Zariski*) if its noetherian and quasi-sober (resp. sober).

Exercise 8.15. Show that $\mathrm{CZpec}(R, \mathscr{C})$ is a quasi-Zariski space whereas $\mathrm{CSpec}(R, \mathscr{C})$ and so $\mathrm{Schpec}(R, \mathscr{C})$ are Zariski spaces.

Theorem 8.12. *Let $(R, \mathscr{C})$ be a Cartier pair. Then,*

$$\beta(R, \mathscr{C}) = \alpha_{\mathrm{CZpec}(R,\mathscr{C})}.$$

Proof. We must show that: if $f\colon \mathrm{CZpec}(R, \mathscr{C}) \longrightarrow X$ in Top is such that $f(\mathfrak{p}) = f(\mathfrak{q})$ whenever $\mathfrak{p} \sim \mathfrak{q}$, then there is a unique continuous function $g\colon \mathrm{CSpec}(R, \mathscr{C}) \longrightarrow X$ factoring f through $\beta(R, \phi)\colon \mathrm{CZpec}(R, \mathscr{C}) \longrightarrow \mathrm{CSpec}(R, \mathscr{C})$.

Recall that,

$$\overline{\{\mathfrak{p}\}}^{\mathrm{CZ}} = \overline{\{\mathfrak{q}\}}^{\mathrm{CZ}} \iff \beta_{\mathfrak{p}}(R, \phi) = \beta_{\mathfrak{q}}(R, \phi)$$

and so the required factorization holds in the category of sets. To further get this factorization in Top, we must show that $Z \subset \mathrm{CSpec}(R, \mathscr{C})$ is closed if so is $\beta^{-1}Z \subset \mathrm{CZpec}(R, \mathscr{C})$. Let $Z \subset \mathrm{CSpec}(R, \mathscr{C})$ be such that $\beta^{-1}Z = V(\mathfrak{a})$ for some $\mathfrak{a} \in I(R, \mathscr{C})$. Note that $\beta_{\mathfrak{p}} \in Z$ if and only if $\beta_{\mathfrak{p}} \in V(\mathfrak{a})$. Since β is a surjection, we conclude that $Z = V_{\mathscr{C}}(\mathfrak{a})$ and so it is closed. $\square$

9. Back to F-regularity and F-rationality

Recall that R is said to be F-regular if (it is F-finite and) any of the following equivalent conditions holds:

(a) The test ideal $\tau(R) = \tau(R, \mathscr{C}_R)$ equals 1.
(b) The element $1 \in R$ is a principal test element: for all $r \in R$ avoiding every minimal prime there is $e \gg 0$ and $\phi \in \mathrm{Hom}_R(F^e_* R, R)$ such that $\phi(F^e_* r) = 1$.
(c) Every proper $\mathscr{C}_R$-ideal is radical and has height 0 (*i.e.* it's an intersection of minimal primes).

Remark 9.1. In fact, the F-regular locus of R is open and given by the complement of $V(\tau(R))$.

We've seen that:

Example 9.2. F-finite regular rings are F-regular and all F-regular rings are F-pure.

For another example,

Exercise 9.1 (Direct summands)**.** Suppose that $\theta\colon R \longrightarrow S$ is a ring homomorphism such that there is $t \in \operatorname{Hom}_R(S,R)$ such that $\theta \circ t = \mathrm{id}$. If S is F-regular then so is R. What if θ is only pure? Conclude that the Veronese algebra

$$\mathcal{k}[x_0^{i_0} \cdots x_n^{i_n} \mid i_0 + \cdots + i_n = d] \subset \mathcal{k}[x_0, \ldots, x_n]$$

and the Segre algebra

$$\mathcal{k}[x_i y_j \mid 1 \leq i \leq m, 1 \leq j \leq n] \subset \mathcal{k}[x_1, \ldots, x_m, y_1, \ldots, y_n]$$

are F-regular. Likewise, if G is a finite group with prime-to-p order acting on $\mathcal{k}[x_1, \ldots, x_n]$ then the ring of invariants $\mathcal{k}[x_1, \ldots, x_n]^G$ is F-regular (what happens if p divides $|G|$?).

Exercise 9.2. Let $R \longrightarrow S$ be a faithfully flat homomorphism of F-finite rings. Show that S is F-regular if so is R.

Exercise 9.3. Suppose that there's $c \in R$ avoiding all minimal primes such that R_c is F-regular. Prove that R is F-regular iff

$$R \longrightarrow F^e_* R \xrightarrow{F^e_* \cdot c} F^e_* R$$

splits for $e \gg 0$.

Exercise 9.4. Let $(R, \mathfrak{m}, \mathcal{k})$ be an F-finite local ring. Show that R is F-regular iff so is $\hat{R}$. More generally, how do $\tau(R)$ and $\tau(\hat{R})$ relate to one another?

If you feel like doing more examples, try this. First, there's a Fedder-type criterion for F-regularity due to Glassbrenner.

Exercise 9.5 (Glassbrenner's criterion for local F-regularity)**.** Let $(S, \mathfrak{n}, \mathcal{k})$ be an F-finite regular local ring and $\mathfrak{a} \subset \mathfrak{n} \subset S$ be an ideal. Consider the quotient local ring $(R = S/\mathfrak{a}, \mathfrak{m} = \mathfrak{n}/\mathfrak{a}, \mathcal{k})$. Prove that the following statements are equivalent.

(a) R is F-regular.
(b) For every $c \in S$ avoiding every minimal prime of $\mathfrak{a}$ there is $e \gg 0$ such that $c(\mathfrak{a}^{[q]} : \mathfrak{a}) \not\subset \mathfrak{n}^{[q]}$.
(c) For some $c \in S$ avoiding every minimal prime of $\mathfrak{a}$ such that R_c is F-regular (*e.g.* regular) there is $e \gg 0$ such that $c(\mathfrak{a}^{[q]} : \mathfrak{a}) \not\subset \mathfrak{n}^{[q]}$.

Exercise 9.6. Let $f \coloneqq x_0^n + \cdots + x_d^n \in S \coloneqq \mathbb{F}_p[x_0, \ldots, x_d]$. Use Glassbrenner's criterion to characterize those triples (p, n, d) for which $R \coloneqq S/f$ is F-regular. Good luck and have fun!

We've also seen that an F-regular ring decomposes as a product of F-regular integral domains. If you don't quite remember we doing this, you should know enough by now to prove it:

Exercise 9.7. Prove that R is F-regular iff it is the product $R = R_1 \times \cdots \times R_k$ of F-regular normal integral domains R_i.

This is why people mostly study F-regularity on integral domains. Our next goal is to show that these are further normal.

For that, it is convenient to recall the following:

Definition 9.3 (Conductor ideal)**.** Let R be a (noetherian) integral domain with field of fractions K. Let $R \subset \bar{R} \subset K$ be the integral closure of R in K. The *conductor ideal* of R is given by

$$\mathfrak{c} := \operatorname{Ann}_R \bar{R}/R = \{r \in R \mid r\bar{R} \subset R\} = \{r \in R \mid \bar{R} \subset (1/r)R \subset K\}$$

Exercise 9.8. Show that:

(a) $V(\mathfrak{c})$ cuts out the non-normal locus of R.
(b) Letting $f\colon \operatorname{Spec} \bar{R} \longrightarrow R$ be the canonical map, prove that $\mathfrak{c}$ is also the image of $\operatorname{Tr}_R\colon f_* f^! R \longrightarrow R$.
(c) $\mathfrak{c} \in I(R, \mathscr{C}_R)$.
(d) $\tau(R) \subset \mathfrak{c}$.

Congrats, you've proved the following:

Proposition 9.4. *An F-regular ring is normal.*

Exercise 9.9. Let R be a normal integral domain. Prove that R is F-regular if and only if for all effective divisors D on $\operatorname{Spec} R$ the canonical map

$$R \xrightarrow{F_*^{e,\#}} F_*^e R \xrightarrow{\subset} F_*^e R(D)$$

splits for all $e \gg 0$.

Exercise 9.10. Show that an F-pure integral domain needn't be normal. However, prove that an F-pure ring (so reduced) is *weakly normal*: if $K = K(R)$ is the total ring of fractions of R and $x \in K$ is such that $x^p \in R$ then $x \in R$. It's a bit harder but still true that F-injective rings are weakly normal.

Open Problem 9.11. Prove that if R is an F-regular integral domain then $\operatorname{Cl} R$ is finitely generated.

More generally,

Open Problem 9.12. Prove that if R is an F-regular integral domain then its Cox ring

$$\bigoplus_{D \in \operatorname{Cl} R} R(D)$$

is noetherian.

Definition 9.5 (Trace ideal of a finite cover)**.** Let $R \subset S$ be a finite extension of rings with corresponding finite cover $f\colon \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ and trace map

$$\operatorname{Tr}_R\colon f_*\omega_f \longrightarrow R$$

where $\omega_f = f^! R$. The *trace ideal* of f is defined to be the image of Tr_R. One writes

$$\tau_f := \operatorname{Tr}_R(f_*\omega_f).$$

Exercise 9.13. Prove that $R \subset S$ splits iff $\tau_f = R$. More generally, show that $V(\tau_f)$ cuts out the locus of points where $R \subset S$ fails to split.

Exercise 9.14. Prove that $\tau_f \in I(R, \mathscr{C}_R)$ and further that $\tau(R) \subset \tau_f$. Congrats, you've proved the following.

Proposition 9.6. *An F-regular ring is a* splinter*, i.e. all finite extensions $R \subset S$ split!*

Definition 9.7. A ring R is said to be a splinter iff all finite extensions $R \subset S$ split.

Open Problem 9.15. Prove that an F-finite splinter ring is F-regular.

Remark 9.8. The F-regularity of splinters have been verified for $\mathbb{Q}$-Gorenstein rings and for those rings for which

$$\bigoplus_{i \in \mathbb{N}} R(-iK_R)$$

is finitely generated. And in dimension ≤ 4.

Exercise 9.16. Prove that an F-finite splinter ring is F-pure and normal.

In fact, one can prove the following.

Theorem 9.9. *A ring R is F-regular if and only if every finite extension splits by an element of $\tau(\omega_f) \subset \omega_f$.*

Open Problem 9.17. Let R be a normal integral domain with field of fractions K. Let R^+ be an absolute integral closure of R, *i.e.* the integral closure of R inside a fixed algebraic closure $\bar{K}$ of K. Define

$$\tau^+ := \bigcap_{R \subset S \subset R^+} \tau_{S/R}$$

where the intersection traverses all finite extensions $R \subset R$ inside R^+. Prove that the inclusion

$$\tau \subset \tau^+$$

is an equality.

Open Problem 9.18. Prove that all elements of $I(R, \mathscr{C}_R)$ are trace ideals.

Theorem 9.10. *F-regular rings are Cohen–Macaulay.*

Proof. We may assume that $(R, \mathfrak{m}, \mathscr{k})$ is a complete local (normal) integral domain. We must prove that $H^i_{\mathfrak{m}}(R) = 0$ for all $i < \dim R$.

Claim 9.11. *There is $0 \neq c \in R$ such that $cH^i_{\mathfrak{m}}(R) = 0$ for all $i < d := \dim R$.*

Proof of claim. In fact, the non-Cohen–Macaulay locus of R is cut out by

$$\bigcap_{i < \dim R} \operatorname{Ann}_R H^i_{\mathfrak{m}}(R)$$

and so we may take c to be a non-zero element there, which exists as R is generically Cohen–Macaulay.

Alternatively, we can do the following. Write $R = S/\mathfrak{p}$ as the quotient of a regular (and so Gorenstein) ring by a prime ideal. Then, we know that $H^i_{\mathfrak{m}}(R)$ is the Matlis dual of $\operatorname{Ext}^{n-i}_S(R, S)$ where $n := \dim S$ (as S-modules). Observe that for all $i < d = n - \operatorname{ht} \mathfrak{p}$ we have

$$S_{\mathfrak{p}} \otimes_S \operatorname{Ext}^{n-i}_S(R, S) = \operatorname{Ext}^{n-i}_{S_{\mathfrak{p}}}(K, S_{\mathfrak{p}}) = \operatorname{Ext}^{\dim S_{\mathfrak{p}} - (i-d)}_{S_{\mathfrak{p}}}(K, S_{\mathfrak{p}}) = 0,$$

where $K = S_{\mathfrak{p}}/\mathfrak{p}S_p$ is the field of fractions of R (which has dimension 0). $\square$

There is $e \gg 0$ and $\phi \in \mathscr{C}_{e,R}$ making the following commutative diagram

$$\begin{tikzcd} R \arrow[r] \arrow[drr, "\mathrm{id}"'] & F^e_* R \arrow[r, "F^e_* \cdot c"] & F^e_* R \arrow[d, "\phi"] \\ & & R \end{tikzcd}$$

Applying the functor $H^i_{\mathfrak{m}}(R)$ with $i < d$ then yields a commutative diagram

$$\begin{tikzcd} H^i_{\mathfrak{m}}(R) \arrow[r] \arrow[drr, "\mathrm{id}"'] & F^e_* H^i_{\mathfrak{m}}(R) \arrow[r, "0"] & F^e_* H^i_{\mathfrak{m}}(R) \arrow[d] \\ & & H^i_{\mathfrak{m}}(R) \end{tikzcd}$$

and so $H^i_{\mathfrak{m}}(R) = 0$; as required. $\square$

Corollary 9.12. *If $R \longrightarrow S$ is a split extension of F-finite rings such that S is regular then R is Cohen–Macaulay.*

Exercise 9.19. Remove the F-finiteness hypothesis from the previous corollary and further that $R \longrightarrow S$ instead of being split it's only pure.[43]

Scholium 9.13. *If R is F-regular then, letting $F \coloneqq H^i_{\mathfrak{m}}(R) \colon H^i_{\mathfrak{m}}(R) \longrightarrow H^i_{\mathfrak{m}}(R)$ the canonical F-module structure on $H^i_{\mathfrak{m}}(R)$, we have that for all $c \in R$ avoiding all minimal primes we have that cF^e is split for $e \gg 0$. This implies that R is a normal Cohen–Macaulay ring for which for all c avoiding minimal primes $cF^e \colon H^d_{\mathfrak{m}}(R) \longrightarrow H^d_{\mathfrak{m}}(R)$ is injective for $e \gg 0$*

Use the above to prove that following.

Exercise 9.20. Let $(R, \mathfrak{m}, \mathcal{k})$ be an Cohen–Macaulay normal local domain with Cartier operator $\kappa \colon F_* R(K_R) \longrightarrow R(K_R)$. Show that R is F-regular iff for all $0 \neq c \in R$ there is $e \gg 0$ such that $\kappa^e c \colon F^e_* R(K_R) \longrightarrow R(K_R)$ is split.

Recall that we had defined a ring R to be F-rational if it is Cohen–Macaulay with and F-regular Cartier operator $\kappa_R \colon F_* \omega_R \longrightarrow \omega_R$.

Exercise 9.21. Let $(R, \mathfrak{m}, \mathcal{k})$ be a Cohen–Macaulay F-finite complete local ring with Cartier operator $\kappa \colon F_* \omega_R \longrightarrow \omega_R$. Prove that:

(a) If $M \subset \omega_R$ is a Cartier submodule then $\mathscr{D}(\omega_R/M) \subset H^d_{\mathfrak{m}}(R)$ is an F-submodule.
(b) If $M \subset H^d_{\mathfrak{m}}(R)$ is an F-submodule, then $\ker(\omega_R \longrightarrow \mathscr{D}(R))$ is a Cartier submodule of ω_R.
(c) Show that those mappings are inverses of one another.
(d) If R is further Gorenstein, use this to write down an explicit bijection between $I(R, \mathscr{C}_R)$ and the F-submodules of $H^d_{\mathfrak{m}}(R)$.
(e) R is F-rational iff $H^d_{\mathfrak{m}}(R)$ is simple as an R-module.

Exercise 9.22. Let $(R, \mathfrak{m}, \mathcal{k})$ be an Cohen–Macaulay local ring with Cartier operator $\kappa \colon F_* \omega_R \longrightarrow \omega_R$. Show that R is F-rational iff for all $0 \neq c \in R$ there is $e \gg 0$ such that $\kappa^e c \colon F^e_* \omega_R \longrightarrow \omega_R$ is surjective.

Exercise* 9.23. Prove that an F-rational ring is necessarily normal.

[43] This is true like this in all characteristics (even mixed).

Corollary 9.14. *F-regular rings are F-rational and the converse holds for Gorenstein rings.*

Remark 9.15. It turns out that splinters are F-rational.

Remark 9.16. In defining F-rationality, we assumed we had a Cartier operator at hand and in particular F-finiteness. A more general definition is often given as follows (and it also implies normality).

Definition 9.17 (F-rationality in general (without Cartier operators))**.** A local ring $(R, \mathfrak{m})$ is F-rational if it is Cohen–Macaulay and for all c avoiding all minimal primes $cF^e\colon H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R)$ is injective for $e \gg 0$.

9.1. **Some few final words on F-signature.** I'd be unacceptable to conclude this introductory course to F-singularities without saying a single word about the F-signature. So here we go.

Definition 9.18 (Generic rank)**.** Let R be an integral domain with field of fractions K. The *(generic) rank* of an f.g. R-module M is

$$\nu(M) \coloneqq \dim_K M \otimes_R K.$$

Remark 9.19. Recall that if $(R, \mathfrak{m}, \mathscr{k})$ is a local ring and M is an f.g. R-module then the *residual rank*

$$\mu(M) \coloneqq \dim_{\mathscr{k}} M \otimes_R \mathscr{k} = \dim_{\mathscr{k}} M/\mathfrak{m}M$$

is the minimal number of generators of M.

Exercise 9.24 (Subaditivity)**.** Let $(R, \mathfrak{m}, \mathscr{k}, K)$ be a local ring and

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of f.g. R-modules. Show that

$$\mu(M) \leq \mu(M') + \mu(M'')$$

whereas

$$\nu(M) = \nu(M') + \nu(M'').$$

Exercise 9.25. Let $(R, \mathfrak{m}, \mathscr{k}, K)$ be a local domain and M be and f.g. R-module. Show that $\nu(M) \leq \mu(M)$ and the equality holds iff M is free of rank $\nu(M) = \mu(M)$.

Definition 9.20 (Free rank)**.** Let $(R, \mathfrak{m}, \mathscr{k}, K)$ be a local domain and M be and f.g. R-module. The free rank of M is

$$\eta(M) \coloneqq \max\{n \in \mathbb{N} \mid \text{there is an exact sequence } M \longrightarrow R^{\oplus n} \longrightarrow 0\}.$$

Exercise 9.26. Show that $\eta(M)$ is well-defined and bounded by $\nu(M)$. That is, show that the set

$$\max\{n \in \mathbb{N} \mid \text{there's an exacts equence } M \longrightarrow R^{\oplus n} \longrightarrow 0\}$$

is non-empty and bounded by $\nu(M)$.

Remark 9.21. Summing up:

$$\eta(M) \leq \nu(M) \leq \mu(M).$$

Exercise 9.27. Let $(R, \mathfrak{m}, \mathscr{k}, K)$ be a local domain and M be and f.g. R-module. Show that:

(a) M is free iff it is torsion free and $\eta(M) = \nu(M)$

(b) $\eta(M) = \mu(M)$ iff M is free.

Exercise 9.28. Let $(R, \mathfrak{m}, \Bbbk, K)$ be a local domain and M be and f.g. R-module. Prove that $\eta(M)$ is the unique $n \in \mathbb{N}$ such that $M \cong R^{\oplus n} \oplus N$ and where N admits no surjections to R (*i.e.* $\operatorname{Hom}_R(N, R)/\operatorname{Hom}(N, \mathfrak{m}) = 0$).

Exercise 9.29. Let $(R, \mathfrak{m}, \Bbbk, K)$ be a local domain and M be and f.g. R-module. Show that

$$M^{\top} \coloneqq \{m \in M \mid \varphi(m) \in \mathfrak{m} \text{ para todo } \phi \in \operatorname{Hom}_R(M, R)\}$$

is a submodule of M that contains $\mathfrak{m}M$. In particular, $M/M^{\top}$ is a $\Bbbk$-module. Show that

$$\eta(M) = \dim_{\Bbbk} M/M^{\top}.$$

Furthere, note that there are inclusions of R-modules

$$\mathfrak{m} \operatorname{Hom}_R(M, R) \subset \operatorname{Hom}_R(M, \mathfrak{m}) \subset \operatorname{Hom}_R(M, R)$$

so that $\operatorname{Hom}_R(M, R)/\operatorname{Hom}_R(M, \mathfrak{m})$ is a $\Bbbk$-module. Prove that

$$\eta(M) = \dim_{\Bbbk} \operatorname{Hom}_R(M, R)/\operatorname{Hom}_R(M, \mathfrak{m}).$$

Next exercise is of the uppermost importance and you must do it.

Exercise 9.30 (Sub-aditividad)**.** Let $(R, \mathfrak{m}, \Bbbk, K)$ be a local domain and

$$0 \to M' \to M \to M'' \to 0.$$

be an exacts equence of f.g. R-modules. Prove that:

$$\eta(M'') \le \eta(M) \le \eta(M') + \mu(M'').$$

Hint: Take $n \in \mathbb{N}$ such that there is a homomorphism of R-modules $\phi\colon M \to R^{\oplus n}$ such that $M' \to M \to R^{\oplus n}$ is surjective. Explain why this induces a commutative diagram os exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "="] & M'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R^{\oplus n} \arrow[r] \arrow[d] & M \arrow[r] & \ker \phi \arrow[r] \arrow[d] & 0 \\
& 0 & & 0 &
\end{tikzcd}$$

so that

$$\eta(M) \le n + \eta(\ker \phi) \le \eta(M') + \mu(M'').$$

Exercise 9.31. Let M be an f.g. module over an integral domain R with field of fractions K. Show that:

(a) there is an exact sequence

$$0 \to R^{\oplus \nu(M)} \to M \to T \to 0$$

such that $K \otimes_R T = 0$, *i.e.* T is torsion. Since T is f.g., therer us $0 \neq r \in R$ tal que $rT = 0$

(b) if M is torsion free (*i.e.* $M \to K \otimes_R M$ is injective), then there is an exact sequence

$$0 \to M \to R^{\oplus \nu(M)} \to T \to 0$$

such that $K \otimes_R T = 0$.

Exercise 9.32. Let $(R, \mathfrak{m}, \mathscr{k}, K)$ be a local domain and M be and f.g. R-module. Show that $\{\mu(F^e_* M)/\nu(F^e_* R)\}_{e>0}$ is a bounded sequence as follows. First, reduce to the case $M = R$. Then proceed by induction on $\lambda(R) \coloneqq \log_q \nu(F^e_* R) \in \mathbb{N}$. If $\lambda(R) = 0$ is a perfect field and there's nothing to do. For the inductive step, consider an exact sequence

$$0 \longrightarrow R^{\oplus p^{\lambda(R)}} \longrightarrow F_* R \longrightarrow T \longrightarrow 0$$

such that $K \otimes_R T = 0$ (see Exercise 9.31). Hence, there is $0 \neq c \in R$ such that $cT = 0$, which means that T is a R/c-module.

Since $\dim R/c < \dim R$, here may want to say that $\lambda(R/c) < \lambda(R)$ to use the inductive hypothesis. However, this isn't possible as R/c may not be an integral domain. To bypass this issue, recall that every f.g. module N over a noetherian ring A admits a filtration

$$0 = N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_n = N$$

such that $N_i/N_{i-1} \cong A/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \operatorname{Spec} A$; see [AM69, Ch. 7, Excercise 18] or [Mat89, Theorem 6.4]. Apply this to T to get a filtration

$$0 = T_0 \subset T_1 \subset T_2 \subset \cdots \subset T_n = T$$

where $T_i/T_{i-1} \cong R/\mathfrak{p}_i$ and $c \in \mathfrak{p}_i \in \operatorname{Spec} R$. In particular, $\dim R/\mathfrak{p}_i \leq \dim R$ and so $\lambda(R/\mathfrak{p}_i) < \lambda(R)$. Use subaditivity and the inductive hypothesis to conclude that:

$$\mu(F^e_* T) \leq \sum_{i=1}^{n} \mu(F^e_* A/\mathfrak{p}_i) \leq C q^{\lambda(R)-1}.$$

for some constant $C = C(R, T)$ and all $e \in \mathbb{N}$.

Use now the esact sequence

$$0 \longrightarrow F^{e-1}_* R^{\oplus p^{\lambda(R)}} \longrightarrow F^e_* R \longrightarrow F^{e-1}_* T \longrightarrow 0$$

to conclude that

$$\mu(F^e_* R) \leq p^{\lambda(R)} \mu(F^{e-1}_* R) + C q^{\lambda(R)-1}$$

for all $e \in \mathbb{N}$. Divide by $\nu(F^e_* R) = q^{\lambda(R)}$ to obtain

$$\frac{\mu(F^e_* R)}{\nu(F^e_* R)} \leq \frac{\mu(F^{e-1}_* R)}{\nu(F^{e-1}_* R)} + \frac{C}{q}$$

for all $e \in \mathbb{N}$. Therefore,

$$\frac{\mu(F^e_* R)}{\nu(F^e_* R)} \leq C + \frac{C}{p} + \frac{C}{p^2} + \cdots + \frac{C}{p^e} \leq \frac{C}{1 - 1/p} \leq 2C.$$

Exercise 9.33. In the setup of Exercise 9.32, note that the above also shows that if T is an f.g. torsion R-module then $\{q\mu(F^e_* T)/\nu(F^e_* R)\}_{e>0}$ is a bounded sequence.

Let $(R, \mathfrak{m}, \mathscr{k}, K)$ be a local domain. Observe that

$$\nu(F^e_* R) = [K^{1/q} : K] = [K^{1/p} : K]^e$$

where $\log_p [K^{1/p} : K] \in \mathbb{N}$. More generally:

Proposition 9.22. *With notation as above,* $\log_p [K^{1/p} : K] = \log_p [\mathscr{k}^{1/p} : \mathscr{k}] + \dim R$, *i.e.*

$$\nu(F^e_* R) = q^{\log_p [\mathscr{k}^{1/p} : \mathscr{k}] + \dim R}$$

Proof. We start by reducing to the complete case. Note that $\hat{R}_{\mathfrak{p}}$ is a field for all minimal prime of $\hat{R}$. Moreover, $\hat{R}_{\mathfrak{p}} \otimes_K K^{1/p} \longrightarrow \hat{R}_{\mathfrak{p}}^{1/p}$ is an isomorphism (*i.e.* $\hat{R}_{\mathfrak{p}}/K$ is separable). In particular,

$$[\hat{R}_{\mathfrak{p}}^{1/p} : \hat{R}_{\mathfrak{p}}] = [K^{1/p} : K]$$

Recall that $\hat{R}_{\mathfrak{p}}$ is then the field of fractions of $\hat{R}/\mathfrak{p}$. Taking $\mathfrak{p}$ such that $\dim \hat{R}/\mathfrak{p} = \dim \hat{R} = \dim R$, we may assume that R is complete.

Next, we reduce to the case in which R is regular. Write R as a finite extension $A \coloneqq \mathscr{k}[\![x_1, \dots, x_{\dim R}]\!]$. Then we have a commutative diagram

$$\begin{tikzcd} A^{1/p} \arrow[r] & R^{1/p} \\ A \arrow[u] \arrow[r] & R \arrow[u] \end{tikzcd}$$

of f.g. A-modules. It follows that $\nu_R(R^{1/p}) = \nu_A(A^{1/p})$. thus, we may assume that $R = A$. I'll let you conclude from here. ☐

Exercise 9.34. Let $(R, \mathfrak{m}, \mathscr{k}, K)$ be a local domain. Show that

$$\eta(F^e_* R) = \dim_{\mathscr{k}} F^e_* R/(F^e_* R)^{\top} = F^e_* R/F^e_* I_e = [\mathscr{k}^{1/q} : \mathscr{k}] \dim_{\mathscr{k}} R/I_e$$

where

$$I_e \coloneqq \{r \in R \mid \phi(F^e_* r) \in \mathfrak{m} \text{ for all } \phi \in \operatorname{Hom}_R(F^e_* R, R)\}.$$

Conclude that

$$0 \leq \frac{\eta(F^e_* R)}{\nu(F^e_* R)} = \frac{\dim_{\mathscr{k}} R/I_e}{q^{\dim R}} \leq 1$$

Theorem 9.23 (Existence of F-signature)**.** *Let $(R, \mathfrak{m}, \mathscr{k}, K)$ be a local domain. The sequence $\{\eta(F^e_* T)/\nu(F^e_* R)\}_{e>0}$ converges.*

Proof. Set $\lambda(R) \coloneqq \log_p \nu(F_* R) \in \mathbb{N}$. Consider an exact sequence

$$0 \longrightarrow R^{\oplus p^{\lambda(R)}} \longrightarrow F_* R \longrightarrow T \longrightarrow 0$$

such that $cT = 0$ for some $0 \neq c \in R$ (see Exercise 9.31). This yields a new exact sequence

$$0 \longrightarrow F^e_* R^{\oplus p^{\lambda(R)}} \longrightarrow F^{e+1}_* R \longrightarrow F^e_* T \longrightarrow 0$$

and so

$$\xi(F^{e+1}_* R) \leq p^{\lambda(R)} \xi(F^e_* R) + \mu(F^e_* T) \leq p^{\lambda(R)} \xi(F^e_* R) + C q^{\lambda(R)-1}$$

for some constant C (independent of e) by using Exercise 9.33. Divide by

$$\nu(F^{e+1}_* R) = p^{(e+1)\lambda(R)} = q^{\lambda(R)} p^{\lambda(R)} = \nu(F^e_* R) p^{\lambda(R)}$$

to get

$$\frac{\xi(F^{e+1}_* R)}{\nu(F^{e+1}_* R)} \leq \frac{\xi(F^e_* R)}{\nu(F^e_* R)} + \frac{C'}{q}$$

where $C' \coloneqq C/p^{\gamma(R)}$ is independent of e. Iterate this inequality to conclude that:

$$\frac{\xi(F^{e+e'}_* R)}{\nu(F^{e+e'}_* R)} \leq \frac{\xi(F^e_* R)}{\nu(F^e_* R)} + \frac{C}{q}\left(1 + \frac{1}{p} + \cdots + \frac{1}{p^{e'-1}}\right) \leq \frac{\xi(F^e_* R)}{\nu(F^e_* R)} + \frac{2C}{q}$$

for all $e, e' \in \mathbb{N}$. Take $\limsup_{e' \to \infty}$ to get

$$\limsup_{e\to\infty} \frac{\xi(F^e_* R)}{\nu(F^e_* R)} \leq \frac{\xi(F^e_* R)}{\nu(F^e_* R)} + \frac{2C}{q}$$

for all e. Take now $\liminf_{e\to\infty}$ to conclude

$$\limsup_{e\to\infty} \frac{\xi(F^e_* R)}{\nu(F^e_* R)} \leq \liminf_{e\to\infty} \frac{\xi(F^e_* R)}{\nu(F^e_* R)}.$$

Since $\{\eta(F^e_* T)/\nu(F^e_* R)\}_{e>0}$ is bounded, we're done. □

Definition 9.24. Let $(R, \mathfrak{m}, \mathcal{k}, K)$ be a local domain. Its *F-signature* is the limit value

$$s(R) \coloneqq \lim_{e\to\infty} \frac{\eta(F^e_* T)}{\nu(F^e_* R)} \in [0,1]$$

Theorem 9.25. *$s(R) = 1$ sii R es regular.*

Proof. The direction "$\Leftarrow$" is an exercise to the reader. For the converse, suppose that R isn't regular. Hence, $F_* R = R^{\oplus a} \oplus M$ for some $M \neq 0$ by Kunz's theorem. In general, we can write

$$F^e_* R = R^{\oplus a_e} \oplus M_e = R^{\oplus a_e} \oplus M^{\oplus b_e} \oplus N_e.$$

where M_e admites no free direct summands and N_e admits no direct summands isomorphic to neither R nor M.

Observe that

$$F^{e+1}_* R = F_* R^{\oplus a_e} \oplus F_* M = R^{\oplus a a_e} \oplus M^{a_e} \oplus F_* M_e.$$

Therefore,

$$b_{e+1} \geq a_e.$$

On the other hand, $\nu(F^e_* R) = q^{\lambda(R)} \geq a_e + b_e$ and so

$$1 \geq \frac{a_e}{q^\lambda} + \frac{b_e}{q^\lambda} \geq \frac{a_e}{q^\lambda} + \frac{a_{e-1}}{p^\lambda p^{(e-1)\lambda}}.$$

Taking the limit $e \to \infty$ yields

$$1 \geq s(R) + s(R)/p^\lambda > s(R).$$

□

Theorem 9.26. *$s(R) > 0$ sii R es F-regular.*

Proof. Note that the sequence

$$\left\{ \frac{\eta(F^e_* R)}{\nu(F^e_* R/\beta(R))} \right\}_{e>0}$$

is bounded as

$$\eta(F^e_* R) \leq \eta(F^e_* R/\beta(R)).$$

Hence, if R isn't F-regular (*i.e.* $\beta(R) \neq 0$) then $s(R) = 0$.

For the converse, □

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