Finite torsors over strongly F-regular singularities†

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Abstract

We discuss an extension to the work by K. Schwede, K. Tucker and myself on the étale fundamental group of strongly F-regular singularities [1]. Concretely, we study the existence of torsors over the regular locus that do not come from restricting a torsor over the whole spectrum. In the abelian case, these torsors naturally relate to the action of Frobenius on local cohomology.

Setup

Let $k = \overline{k}$ be our groundfield, char $k = p > 0$. Let $(R, \mathfrak{m}, k)$ be a str. F-regular strictly local domain. Set $X = \mathrm{Spec}(R)$, $U = X_{\mathrm{reg}}$ and $Z = X_{\mathrm{sing}}$.

Abelian torsors

For G abelian group-scheme, let

$$\mathrm{Ob}_X(G) := \mathrm{coker}\big(H^1(X, G) \to H^1(U, G)\big)$$

measure the obstructions to extend everywhere a G-torsor over U. How large can $\mathrm{Ob}_X(G)$ be?

- For $G = \mathbb{Z}/\ell\mathbb{Z}$, $p \nmid \ell$, see [1].
- For $G = \mathbb{Z}/p^e\mathbb{Z}$, Artin-Schreier Theory gives:
  $$\mathrm{Ob}_X(\mathbb{Z}/p^e\mathbb{Z}) = H^2_Z(R)^{F^e} = 0 \quad c.f. \ [3].$$
- For $G = \boldsymbol{\mu}_{p^e}$, Kummer Theory gives:
  $$\mathrm{Ob}_X(\boldsymbol{\mu}_{p^e}) \leftrightarrow \Big\{ \big(\mathcal{L}, \mathcal{O}_U \xrightarrow{\cong} \mathcal{L}^{p^e}\big) \mid 0 \neq \mathcal{L} \in \mathrm{Pic}\ U \Big\}.$$
- For $G = \boldsymbol{\alpha}_{p^e}$, we have:
  $$\mathrm{Ob}_X(\boldsymbol{\alpha}_{p^e}) = \ker\big(H^2_Z(R) \xrightarrow{F^e} H^2_Z(R)\big) = 0.$$

Conclusion: Need to study $\mathrm{Ob}_X(\boldsymbol{\mu}_{p^e})$ and cyclic covers over X. Key:

Generalized Transformation Rule

Let $(A, \mathfrak{a}) \subset (B, \mathfrak{b})$ be a finite local extension. Suppose $\exists T \in \mathrm{Hom}_A(B, A)$ s.t.: $B \cdot T = \mathrm{Hom}_A(B, A)$, T is onto and $T(\mathfrak{b}) \subset \mathfrak{a}$. Then

$$\big[k(\mathfrak{b}) : k(\mathfrak{a})\big] \cdot s(B) = \dim_{K(A)} B_{K(A)} \cdot s(A).$$

So, B is a str. F-regular if (and only if) A is so. Same holds for F-purity.

How to apply the trans. rule?

- Let $h : V \to U$ be a connected G-torsor over U. By taking int. closure of h, we get a G-quotient $(R, \mathfrak{m}, k) \subset (S, \mathfrak{n}, k)$, a G-torsor in codimension-1.
- Where to get that T from? From the theory of integrals for Hopf algebras!
- $\mathrm{Tr}_{S/R}$ is onto since R is splinter, $S \cdot \mathrm{Tr}_{S/R} = \mathrm{Hom}_R(S, R)$ holds in codim-1 so everywhere.
- $\mathrm{Tr}_{S/R}(\mathfrak{n}) \subset \mathfrak{m}$ is rather subtle and not always true. However, it holds for Veronese-type cyclic covers!

Integrals for Hopf alg's and Traces

Finite Hopf alg's (*e.g.* $\mathcal{O}(G)^{\vee}$) come equipped with a special element, unique up to k-scaling, called *integral*. Using this integral and the given action, any G-quotient $S^G \subset S$ can be provided with a *trace map* $\mathrm{Tr}_{S/S^G} : S \to S^G$.

Trace characterizes torsor-ness

$S^G \subset S$ is a G-torsor iff it is locally free of rank $o(G)$ and $\mathrm{Tr}_{S/S^G} \cdot S = \mathrm{Hom}_{S^G}(S, S^G)$.

F-signature goes up under Veronese-type cyclic covers

Main result: Existence of Maximal Cover

There exists a *nice* finite cover $(R, \mathfrak{m}) \subset (R^\star, \mathfrak{m}^\star)$ such that: $R^\star$ is str. F-regular and any trigonalizable or nilpotent torsor over its own regular locus is a torsor everywhere, *e.g.* the local abelian Nori fundamental group-scheme of $R^\star$ is trivial, as defined in [2].

If $\mathcal{L} \in \mathrm{Pic}\ U$ with index n, the Veronese-type cover $C(\mathcal{L}) := \bigoplus_{i=0}^{n-1} H^0(U, \mathcal{L}^i) \supset R$ satisfies $\mathrm{Tr}_{C/R}(\mathfrak{n}) \subset \mathfrak{m}$, so $s(C) = n \cdot s(R)$. In fact, C would be F-pure if R were only assumed F-pure. By taking $\mathcal{L} = \omega_U$: **canonical covers of str. F-reg. (r. F-pure) singularities are str. F-reg. (r. F-pure), even if $p \mid n$.** Moreover, if $(R, \mathfrak{m}) \subset (S, \mathfrak{n})$ is a $\boldsymbol{\mu}_{p^e}$-torsor over U but not everywhere, there must be a nontrivial Veronese-type cover over R. One iterates this until $s(R)$ gets exhausted, explaining the abelian case.

On the proof of the trans. rule

Letting $q : \mathrm{Spec} B \to \mathrm{Spec} A$, we go through:

$$\begin{aligned}
&\big[k(\mathfrak{b}) : k(\mathfrak{a})\big] \cdot s(B) \\
&= \lim_{e \to \infty} \frac{\big[k(\mathfrak{b}) : k(\mathfrak{a})\big]}{p^{e\delta}} \lambda_B \left(\frac{\mathrm{Hom}_B(F^e_* B, B)}{\mathrm{Hom}_B(F^e_* B, \mathfrak{b})} \right) \\
&= \lim_{e \to \infty} \frac{1}{p^{e\delta}} \lambda_A \Big(q_* \mathrm{Hom}_B(F^e_* B, B) / q_* \mathrm{Hom}_B(F^e_* B, \mathfrak{b}) \Big) \\
&= \lim_{e \to \infty} \frac{1}{p^{e\delta}} \lambda_A \Big(\mathrm{Hom}_A(q_* F^e_* B, A) / \mathrm{Hom}_A(q_* F^e_* B, \mathfrak{a}) \Big) \\
&= \lim_{e \to \infty} \frac{1}{p^{e\delta}} \lambda_A \Big(\mathrm{Hom}_A(F^e_* q_* B, A) / \mathrm{Hom}_A(F^e_* q_* B, \mathfrak{a}) \Big) \\
&= \dim_{K(A)} B_{K(A)} \cdot s(A).
\end{aligned}$$

It makes explicit the use of Grothendieck duality for q. Last step is [4, Theorem 4.11].

Applications to the Picard group

Since $1 \geq s\big(C(\mathcal{L})\big) = n \cdot s(R)$, we get right away:

Boundedness of the torsion

The torsion of Pic U is bounded by $1/s(R)$. In particular, Pic U is torsion-free if $s(R) > 1/2$.

By taking affine cones:

Let Y be a globally F-regular variety, then the torsion of Cl Y is bounded by the reciprocal of the F-signature of any section ring of Y.

Beyond the solvable case

I am grateful to A. Stäbler for bringing to my attention the recent classification of all (rank-1) simple finite group-schemes, see [5] for a nice, brief account. Letting $\varrho_X(G) : \check{H}^1(X_{\mathrm{ft}}, G) \to \check{H}^1(U_{\mathrm{ft}}, G)$ for a G in this list, the following questions are in order:

- For which G is $\varrho_X(G)$ surjective?
- For G with non-surjective $\varrho_X(G)$: if $R \subset S$ is a torsor over U but not everywhere, does $\mathrm{Tr}_{S/R}(\mathfrak{n}) \subset \mathfrak{m}$ hold?
- For a given G, for which type of (F-)singularity X, if any, is $\varrho_X(G)$ naturally surjective?

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