Fundamental groups of F-regular singularities via F-signature[†]

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Abstract

We prove that the étale fundamental group of a strongly F-regular singularity is finite, analogous to results of Xu and Greb-Kebekus-Peternell for KLT singularities in characteristic zero. In fact our result is effective, we show that the reciprocal of the F-signature of the singularity gives a bound on the size of this fundamental group. To prove these results and their corollaries, we develop new transformation rules for the F-signature under finite étale-in-codimension-one extensions. As another consequence of these transformation rules, we also obtain purity of the branch locus over rings with mild singularities.

Fundamental group of a strongly F-regular singularity

J. Kollár asked if $(0 \in X)$ is the germ of a KLT singularity, is $\pi_1(X \setminus \{0\})$ is finite. [1, Question 26]

- C. Xu showed that this holds for the étale local fundamental group [2].
- Greb-Kebekus-Peternell proved the finiteness of the étale fundamental groups of the regular locus of KLT singularities [3].

However, we know that

$$\left\{\begin{array}{c}\text{KLT}\\ \text{singularities}\\ \text{in char. } 0\end{array}\right\} \rightsquigarrow \left\{\begin{array}{c}F\text{-regular}\\ \text{singularities}\\ \text{in char. } p\end{array}\right\}$$

Hence it is natural to ask if the same results hold for F-regular singularities.

Theorem A

Let $(R, \mathfrak{m}, k)$ be a normal F-finite and strongly F-regular strictly Henselian local domain of prime char. $p > 0$, with dimension $d \geq 2$. Then the étale fundamental group of its punctured spectrum is finite with order at most $1/s(R)$ and prime to p. The same also holds for $\pi_1^{\text{ét}}(\operatorname{Spec}(R) \setminus Z)$ where $Z \subseteq \operatorname{Spec} R$ has codimension ≥ 2.

F-signature goes up under the presence of ramification

Our proof of Theorem A relies on a study of the growth of the F-signature under étale-in-codimension-one local finite extensions. In fact, one has a transformation rule for this invariant under this kind of extensions:

Transformation rule for F-signature

Let $(R, \mathfrak{m}, k) \subseteq (S, \mathfrak{n}, \ell)$ be a finite local extension of F-finite d-dimensional normal domains in char. $p > 0$, with extension of fraction fields $K \subseteq L$. Suppose $R \subseteq S$ is étale in codimension 1, and that R is strongly F-regular. Then if one writes $S = R^{\oplus f} \oplus M$ as a decomposition of R-modules so that M has no nonzero free direct summands, then $f = [\ell : k] \geq 1$ and the following equality holds:

$$s(S) = \frac{[L : K]}{[\ell : k]} \cdot s(R).$$

Therefore, if the extension is not étale everywhere, then $s(S) \geq 2s(R)$

On the proof of the trans. rule

Write $S^{1/p^e} = S^{\oplus a_e(S)} \oplus N_e$ a decomposition of S-modules, where N_e doesn't admit a free direct summand. Hence one also has a decomposition of R-modules $S^{1/p^e} = R^{f \cdot a_e(S)} \oplus M^{\oplus a_e(S)} \oplus N_e$. Thus, if one defines b_e as the maximal rank of a free R-module appearing in a direct sum decomposition of S^{1/p^e}, then $b_e \geq f \cdot a_e(S)$. To get an equality one has to ensure that there are no direct free R-summands coming from N_e. This is actually what we do. After that one uses [4, Theorem 4.11]. For last assertion one goes into the **strong tameness on ramification that F-regularity imposes**. Indeed, one has that ℓ/k is separable and $[\ell : k] \mid [L : K]$. Then one notices $[\ell : k] \neq [L : K]$ follows from purity of the branch locus for finite faithfully flat morphisms.

F-signature

The F-signature; explicitly introduced in [5], measures how many different ways $R \hookrightarrow F^e_* R$ splits as e goes to infinity. More precisely, if R has perfect residue field and $F^e_* R = R^{\oplus a_e} \oplus M$ as an R-module, where M has no free R-summands, then $s(R) = \lim_{e \to \infty} a_e / p^{e \dim R}$. Here are three quick facts:

- The limit $s(R) \in [0, 1]$ exists [4].
- $s(R) > 0$ iff R is strongly F-regular [6].
- $s(R) = 1$ iff R is regular [5].

Note on pairs: One also obtains analogous results in the context of pairs. Considering F-regular pairs (R, Δ) and the divisor upstairs $\pi^*\Delta - \text{Ram} \geq 0$.

Purity of branch locus

As a notable consequence, one obtains purity of the branch locus for mild singularites, with F-signature more than one-half.

Corollary

Suppose $Y \to X$ is a finite dominant map of F-finite normal integral schemes. If $s(\mathcal{O}_{X,x}) > 1/2$ for all $x \in X$ then the branch locus of $Y \to X$ has no irreducible components of codimension ≥ 2, in other words it is a divisor.

On the proof of Theorem A

Thus we get that F-signature gives a maximum size for an étale-in-codimension-one extension (since residue field extensions are trivial due to tameness on ramification and $k = k^{\text{sep}}$), then there would exists a maximal étale-in-codimension-one extension, a "universal cover". The generic degree of such a cover equals the order of the group, which allows us to see how $1/s(R)$ is a bound and why p doesn't divide it.

An example of a global corollary

Theorem B

Suppose (X, Δ) is a globally F-regular projective pair over an algebraically closed field of char. $p > 0$. There is a number n such that every finite separable cover $\pi : Y \to X$ with $\pi^*\Delta - \text{Ram} \geq 0$ has generic rank $[K(Y) : K(X)] \leq n$.

This follows by working out for pairs the transformation rule of F-signature and taking cones. Other global corollaries are obtained in the same fashion.

Example of quotient singularities

In [5] the F-signature of the 2-dim. rat. dble-pts

$$s\big((A_n), (D_n), (E_6), (E_7), (E_8)\big) = |G|^{-1}$$

From our perspective, this is reinterpreted by noticing the existence of a smooth cover S, then a "universal cover", so that

$$\text{gen. degree} = [K(S) : K(R)] = |\pi_1| = s(R)^{-1},$$

the bound is realized.

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